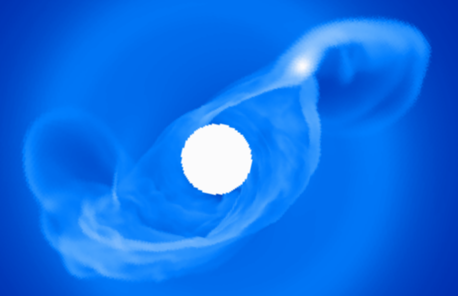
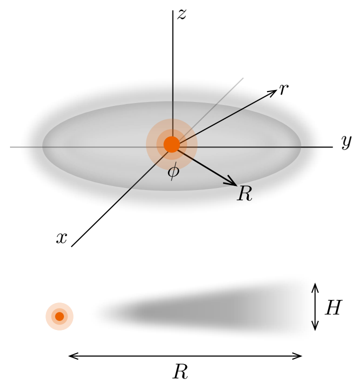
accretion disks



Classical hydrodynamics of accretion disks



Geometry: cylindrical coordinates with a surface density defined as the vertical integration of the density, $\Sigma = \Sigma(R, t) = \int \rho(\vec{\mathbf{r}}, t) dz$.

Basic equations: a) continuity equation $\frac{\partial \Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R \Sigma v_R) = 0$ [ceq]; b) Navier-Stokes equations: $\frac{\partial \vec{\mathbf{v}}}{\partial t} + (\vec{\mathbf{v}} \cdot \vec{\nabla})\vec{\mathbf{v}} = -\frac{1}{\rho}\left(\vec{\nabla} P + \vec{\nabla} \cdot \overleftrightarrow{\sigma}\right) - \vec{\nabla}\Phi$.

Viscosity: the stress tensor, in its more relevant component of shear stress $\sigma_{R\phi}$, can be written as $\sigma_{R\phi} = \rho \nu R \frac{d\Omega}{dR}$, with $\Omega = v_\phi / R$ and ν the kinematic viscosity ($\mathscr{D}[\nu] = L^2/T$)

Simplifications: we assume that the disk is thin, $H/R \ll 1$, and that $v_R \ll c_s \ll v_\phi$. We also assume that the disk is not self-gravitating, that is, the central massive body dominates gravity and then, $\Phi = -GM/r$.

Speed of sound: $c_s^2 = \frac{\text{"stress"}}{\text{"inertia"}} = \gamma \frac{dP}{d\rho}$. For slow perturbations, we can take the approximation $\gamma \sim 1$, that is, that the process is isothermal, but in general, $c_s = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\frac{\gamma N k_B T(R)}{\rho}}$.

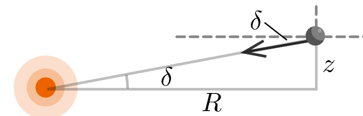
Radial component and the 0th-order solution, Keplerian disks: the radial component of the Navier-Stokes equations is $\cancel{\frac{\partial v_R}{\partial t}} + \cancel{v_R \frac{\partial v_R}{\partial R}} - \frac{v_\phi^2}{R} = -\cancel{\frac{1}{\rho}\frac{\partial P}{\partial R}} - \frac{\partial \Phi}{\partial R}$ [NSR], where since $v_R \ll v_\phi$, the first two terms on the lhs are negligible, and the first term of the rhs is negligible since $\frac{1}{\rho}\frac{\partial P}{\partial R} = \frac{1}{\rho}\frac{\partial P}{\partial \rho}\frac{d\rho}{dR} = \frac{c_s^2}{\rho}\frac{\partial \rho}{\partial R} \sim \frac{c_s^2}{R}$, but $c_s \ll v_\phi$. The remaining terms determine a Keplerian orbit, and therefore, we have a Keplerian disk, with $v_\phi^2 = v_k^2 = GM/R \implies v_\phi = \sqrt{GM/R} \implies \Omega_k^2 = GM/R^3$, and specific angular momentum is $\sqrt{GMR}$. We are using the approximation of order of magnitude that $T = \text{const}$, which is not true (below we derive a relation $T(R)$).

First-order correction: pressure forces in the radial direction: small departures from the Keplerian rotation can occur, e.g., when there are small solid bodies within the disk. Let us assume that $\rho \propto R^{-\beta}$; now, the pressure term is not negligible. Then, NSR becomes $\frac{v_\phi^2}{R} = \frac{1}{\rho}\frac{\partial P}{\partial R} + \frac{GM}{R^2}$, and the first term of the rhs is $\frac{1}{\rho}\frac{\partial \rho}{\partial R}\frac{\partial P}{\partial \rho}$ $= -\beta R^{-1} c_s^2$. $\frac{v_\phi^2}{R} = \frac{-\beta c_s^2}{R} + \frac{GM}{R^2} \implies v_\phi = \sqrt{\frac{GM}{R}\left(1 - \beta \frac{c_s^2}{v_k^2}\right)}$.

z-component of the Navier-Stokes equation: for a thin disk (small z), $0 = -\frac{1}{\rho}\frac{\partial P}{\partial z} - \frac{\partial \Phi}{\partial z}$

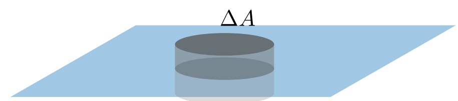
$\implies \frac{1}{\rho}\frac{dP}{d\rho}\frac{d\rho}{dz} = -\frac{\partial \Phi}{\partial z} \implies \frac{c_s^2}{\rho}\frac{d\rho}{dz} = -\frac{GMz}{R^3} = -\Omega_k^2 z \implies \ln\left(\frac{\rho}{\rho_0}\right) = \frac{-\Omega_k^2 z^2}{2c_s^2}$

$\implies \rho = \rho_0 e^{-\frac{\Omega_k^2 z^2}{2c_s^2}}$. We define $\Omega_k^2/c_s^2 := 1/H^2$ as the *thickness*, and then we have $\frac{H}{R} = \frac{c_s}{v_k} \ll 1$, $\therefore$ thin disk $\implies$ disk rotating highly supersonic.



$$F_{Gz} = -\frac{GMz}{(R^2+z^2)^{3/2}} = -\frac{-GMz}{R^3}$$

Limiting case: vertical gravitational field is dominated by self-gravity: $F_{\text{self-grav}} = -2\pi G\Sigma$ ($\mathscr{D}[F] = F/M$). Result: $\rho(z) = \rho_0 \frac{1}{\cosh^2(z/H_{\text{self-grav}})}$ where $H_{\text{self-grav}} := c_s^2/(\pi G \Sigma)$.



$$\oint \vec{\mathbf{g}} \cdot d\vec{\mathbf{A}} = 4\pi G M_{\text{enc}}$$
$$M_{\text{enc}} = \Sigma \Delta A$$

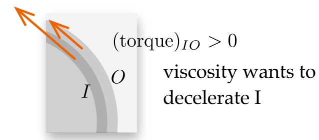
Angular momentum transport

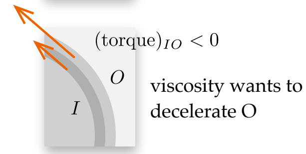
ϕ-component of the Navier Stokes equations: first, we write $\vec{\mathbf{v}} = v_R\hat{\mathbf{R}} + v_\phi\hat{\phi}$, $\vec{\nabla} = \hat{\mathbf{R}}\partial_R + \hat{\phi}(1/R)\partial_\phi$; $\partial_\phi\hat{\mathbf{R}} = \hat{\phi}$, $\partial_\phi\hat{\phi} = -\hat{\mathbf{R}}$. Then, with that, we find $[(\vec{\mathbf{v}}\cdot\vec{\nabla})\vec{\mathbf{v}}]_\phi = v_R\partial_R v_\phi + v_\phi v_R/R + \cancel{(v_\phi v_R/R)\partial_\phi v_\phi}$ (last term is zero because of axisymmetry). The axisymmetry also implies $\nabla P, \nabla\Phi$ do not contribute. Then,
$\Sigma\left(\frac{\partial v_\phi}{\partial t} + \frac{v_R v_\phi}{R} + v_R\frac{\partial v_\phi}{\partial R}\right) = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 T_{R\phi})$ [NSPhi]

Angular momentum transport: NSPhi $\times R$, lhs: $\Sigma v_R v_\phi + \Sigma R\frac{\partial v_\phi}{\partial t} + R\Sigma v_R\frac{\partial v_\phi}{\partial R}$, adding and substracting "weird terms" (underlined), we have
$\underline{-\Sigma v_R v_\phi} - \underline{\frac{\partial \Sigma}{\partial R} v_R R v_\phi} - \underline{\Sigma R v_\phi\frac{\partial v_R}{\partial R}} + \Sigma R\frac{\partial v_\phi}{\partial t} + 2v_R v_\phi\Sigma + R\Sigma v_\phi\frac{\partial v_R}{\partial R} + R v_R v_\phi\frac{\partial \Sigma}{\partial R} + R\Sigma v_R\frac{\partial v_\phi}{\partial R}$. Using ceq , we have $\left[\frac{\partial\Sigma}{\partial t} = -\frac{1}{R}\Sigma v_R - \frac{\partial \Sigma}{\partial R}v_R - \Sigma\frac{\partial v_R}{\partial R}\right]\cdot R v_\phi$. Keeping in mind that since the equation describes fields, $\partial R/\partial t = 0$, the remaining terms can be written more nicely because $\frac{\partial}{\partial t}(\Sigma R v_\phi) = R v_\phi\frac{\partial\Sigma}{\partial t} + \Sigma R\frac{\partial v_\phi}{\partial t}$ and $\frac{1}{R}\frac{\partial}{\partial R}(R^2 v_R\Sigma v_\phi) = \frac{1}{R}2R v_R v_\phi\Sigma + R\Sigma\frac{\partial v_R}{\partial R}v_\phi + R\frac{\partial\Sigma}{\partial R}v_R v_\phi + R\Sigma v_R\frac{\partial v_\phi}{\partial R}$. Therefore,
$\frac{\partial}{\partial t}(\Sigma R v_\phi) + \frac{1}{R}\frac{\partial}{\partial R}(R v_R\Sigma R v_\phi) = \frac{1}{R}\frac{\partial}{\partial R}(R^2 T_{R\phi})$ [ameq]. The lhs of this equation is the ϕ-component of the Lagrangian derivative $[\partial_t\vec{\ell} + (\vec{\mathbf{v}}\cdot\vec{\nabla})\vec{\ell}]_\phi$ of the specific angular momentum $\vec{\ell} = \Sigma R v_\phi\hat{\phi}$, keeping in mind the axisymmetry and that $v_R \ll c_s \ll v_\phi \implies \partial_R v_R \to 0$. The rhs is the stress tensor (that carries viscosity) integrated in the z-direction. This equation can be interpreted as a statement that angular momentum transport occurs because of the torque done by viscosity.

The case of Keplerian disks: The Keplerian relations $v_\phi = \sqrt{GM/R}$, $\Omega = v_\phi/R$ imply for their radial derivatives that $\Omega' = -\frac{3}{2}\sqrt{GM}R^{-5/2}$ and $(Rv_\phi)' = \frac{1}{2}\sqrt{GM}R^{-1/2}$. The stress tensor is $T_{R\phi} = \nu\Sigma R\Omega'$.
$\implies \underbrace{\frac{\partial}{\partial t}(\Sigma R^2\Omega)}_{(a)} + \underbrace{\frac{1}{R}\frac{\partial}{\partial R}(\Sigma v_R R^3\Omega)}_{(b)} = \frac{1}{R}\frac{\partial}{\partial R}(\nu\Sigma R^3\Omega')$. The term (b) can be re-written as (b) $= (\Omega/R)\partial_R(\Sigma v_R R^2 R) = (\Omega/R)\cdot 2R\Sigma v_R R + (\Omega/R)R^2\partial_R(\Sigma v_R R)$. The second term of (b) and the term (a) make up the lhs of the continuity equation, and therefore are 0. We are left out with
$2\Omega R\Sigma v_R = \frac{1}{R}\frac{\partial}{\partial R}(\nu\Sigma R^3\Omega')$. But $R\Sigma\partial_R(R^2\Omega) = R\Sigma(2R)\Omega$, and then we can solve for v_R:

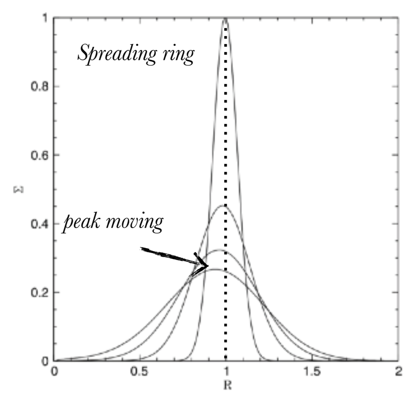
$$\implies v_R = \frac{\partial_R(\nu\Sigma R^3\Omega')}{R\Sigma\partial_R(R v_\phi)} = \frac{\partial_R\left(\nu\Sigma R^3 \cdot \frac{-3}{2}\sqrt{GM}R^{-5/2}\right)}{R\Sigma \cdot \frac{1}{2}\sqrt{GM}R^{-1/2}} = \frac{-3}{\Sigma R^{1/2}}\frac{\partial}{\partial R}(\nu\Sigma R^{1/2}).$$ Inserting in the continuity equation, we finally have $\frac{\partial\Sigma}{\partial t} = \frac{3}{R}\frac{\partial}{\partial R}\left[R^{1/2}\frac{\partial}{\partial R}(\nu\Sigma R^{1/2})\right]$ [Kdiskev]. This is a differential equation that intends to describe how the disk evolves, how the distribution of Σ is with R and t.



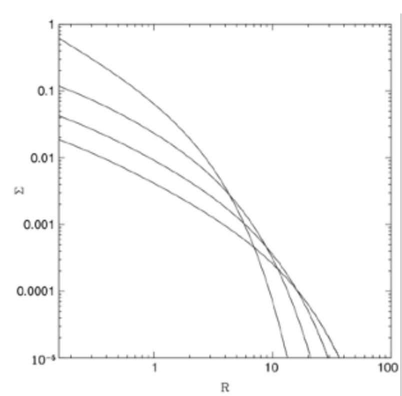


Dependency of the angular momentum transport: the torque produced by an inner annulus (I) onto an outer annulus (O) is $G(R) = -2\pi R^2 T_{R\phi} = -2\pi\nu\Sigma R^3\Omega'$ (the 2π comes from the anular segment integration; if $\Omega = \text{const}$, i.e. solid rotation, there is no angular momentum transport by viscosity). For a Keplerian disk, $G(R) = 2\pi R^2\nu\Sigma R^3\sqrt{GM}/R^{5/2} = 2\pi\nu\Sigma\sqrt{GMR}$, which implies an infall of matter and an outflux of angular momentum. So, hydrodynamics predict that $T_{R\phi} \propto \Omega'$. The correct derivation with kinetic theory is given in Clarke & Pringle 2004 MNRAS 351 1187. They consider a fluid in both a linear and a circular flow, and consider two points ("emitter" of particles —due to collisions— and "receiver" of particles). Those two points are not in rest one respect to the other (due to the $\vec{v}$ field), and then calculate the momentum transmission in an inertial frame. In the case of a linear flow, they get a shear tensor proportional to the gradient of velocity (as expected from basic theory), and for a circular flow, the tensor is proportional to the gradient of the angular velocity (due to "Coriolis effects"). Previous articles on that matter calculated that kinetic theory predicted $T_{R\phi} \propto (R^2\Omega)'$ (gradient of the specific angular momentum) on the ground of isotropy, but ignored the correct Coriolis effect, that effectively gives a preferred direction to angular momentum transport. Under that incorrect prediction, for a Keplerian disk $G(R) \propto -\nabla(R^2\Omega) \implies -\partial_R(R^2\Omega) = -\partial_R(\sqrt{GMR}) = -\sqrt{GM}R^{-1/2}/2$, which implies no accretion!

Analytical solutions for accretion disks



The spreading ring: Kdiskev with constant ν and initial conditions of $\Sigma(R, t = 0) = \frac{m}{2\pi R_0}\delta(R - R_0)$ represents an initial configuration of a ring of radius R_0 and mass m. The solution for the time evolution is of the form $\Sigma(x, t) \propto x^{-1/4}\tau^{-1}e^{-(1+x^2)/\tau} \times \text{Bessel}(2x/\tau)$, with $\tau := 12\nu t/R_0^2$, which implies that the typical timescale of the viscosity is $t_\nu \sim R^2/\nu$. That is a general result. Features: a) the ring is spreading in both sides due to the angular momentum conservation; b) the transition of in/out -flow varies with time, so, eventually, most of the matter is accreted.



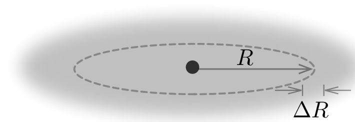
Self-similar solution: in general, $\nu \propto R^b$. Let's consider $\nu \propto R$, and an initial density profile such that inside a radius R_1 we have constant accretion and outside, it decreases exponentially. The solution in this case is of the form $\Sigma(R, t) \propto \frac{T^{-3/2}}{\nu(R)}e^{-\frac{R/R_1}{T}}$, with $T := 1 + t/(R_1^2/[3\nu(R_1)])$, and so, again we have $t_\nu \sim R^2/\nu$; observationally, if we measure the fraction of young stars with disks in clusters, we can have an estimate for t_ν, and if we resolve for R, we can give an estimate for ν. The accretion rate is $\dot{M}(R, t) \propto T^{-3/2}\left[1 - \frac{2R/R_1}{T}\right]e^{-\frac{R/R_1}{T}}$. Features: a) it spread outwards at first; b) most of the mass is eventually accreted; c) $R_{\text{transition}} \propto t$; d) with $T \to \infty \implies \dot{M} \to 0$. [Notation warning: only in this section T is not temperature]

Stationary state solution: ($\dot{M}$ = const) The continuity equation becomes $\dot{M} = -2\pi R v_R \Sigma$. ameq integrated in z implies that $\underbrace{\frac{\partial}{\partial t}\int \Sigma R\, dR\, d\phi R^2\Omega}_{j} + \underbrace{\int \frac{1}{\cancel{R}}\frac{\partial}{\partial R}(\Sigma v_R R^3\Omega)\cancel{R}\, dR\, d\phi}_{(a)} = \underbrace{\int \frac{1}{\cancel{R}}\frac{\partial}{\partial R}(\nu\Sigma R^3\Omega')\cancel{R}\, dR\, d\phi}_{(b)}$. After carrying out the integration, (a) can be rewritten as $\Sigma v_R R^3\Omega \cdot 2\pi = \Sigma 2\pi R v_R R^2\Omega = \dot{M}R^2\Omega$. Also, (b) $= 2\pi\nu\Sigma R^3\Omega'$ $= -2\pi\nu\Sigma R^3 \cdot \frac{3}{2}\sqrt{GM}R^{-5/2} = -3\pi\nu\Sigma\sqrt{GMR} = -3\pi\nu\Sigma R^2\Omega$. Then, we have $\dot{J} = \dot{M}R^2\Omega - 3\pi\nu\Sigma R^2\Omega$.

No-torque assumption: at a radius R_{in}, the angular velocity flattens $\implies \nabla\Omega = 0$, due to a boundary layer that connects the star with the disk. Therefore, as the torque goes to zero, viscosity goes to zero. Then, $\dot{J} = \dot{M}(\Omega R^2)_{\text{in}} = \dot{M}\sqrt{GMR_{\text{in}}}$. Then, we have $\dot{M}\sqrt{GMR_{\text{in}}} = \dot{M}\sqrt{GMR} - 3\nu\Sigma\sqrt{GMR}$ $\implies -\dot{M}\sqrt{R_{\text{in}}/R} + \dot{M} = 3\pi\nu\Sigma \implies 3\pi\nu\Sigma = \dot{M}\left(1 - \sqrt{R_{\text{in}}/R}\right)$. For $R \gg R_{\text{in}}$, we have finally $3\pi\nu\Sigma = \dot{M}$ and so, $\dot{M} \propto \nu$.

Accretion disks energetics

How much energy is radiated by the disk: Gravitational potential energy is dissipated by viscous forces. (Total torque exerted by an annulus on other one with higher R) = "ΔG" $= G(R - \Delta R/2) - G(R + \Delta R/2) = -\frac{\partial G}{\partial R}\Delta R$.



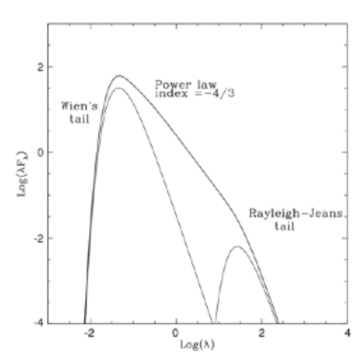
The power is $\frac{\Delta W}{\Delta t} = \frac{1}{\Delta t}\int(\text{torque})d\theta \approx \Delta G\Omega \implies$

(power) $= -\frac{\partial G}{\partial R}\Delta R\Omega = -\left(\underbrace{\frac{\partial}{\partial R}(GR)}_{(a)} - \underbrace{G\Omega'}_{(b)}\right)\Delta R$. The term (a) is the energy *transport* due to viscosity, and when integrated vanishes except for the energy transported out of the boundaries; the term (b) is the energy *dissipation* due to viscosity. The power dissipated per unit area is by both sides of the disk is $D(R) = -\frac{G\Omega'}{2\pi R(\Delta R)} = \frac{\cancel{2\pi} R^{\cancel{2}}\nu R\Sigma\Omega'^2}{\cancel{2\pi R}} = \nu\Sigma(R\Omega')^2$ (see stationary state solution) and for a Keplerian disk, $= \frac{9}{4}\nu\Sigma\Omega^2$. The luminosity emitted is

$$L_{\text{disk}} = \int_{R_{\text{in}}}^{\infty} 2\pi R D(R) dR = \int_{R_{\text{in}}}^{\infty} 2\pi R \cdot \frac{\dot{M}}{3\pi\nu}\left(1 - \sqrt{\frac{R_{\text{in}}}{R}}\right)\nu\frac{9}{4}GM\frac{R^2}{R^5}dR = \cdots = \frac{1}{2}\frac{GM\dot{M}}{R_{\text{in}}},$$

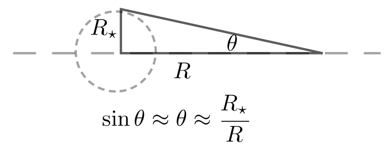
that is, half of the potential energy is dissipated by viscosity.

Radial temperature profile: the power dissipated by an annulus is $2\pi R D(R)\Delta R = \frac{3GM\dot{M}}{2R^2}\Delta R$. If the power is radiated as a blackbody, $D(R) = 2\sigma_{SB}T_s^4$ (2: two sides) $\implies T_s^4 = \frac{3GM\dot{M}}{8\pi\sigma_{SB}R^3}\left(1 - \sqrt{\frac{R_{\text{in}}}{R}}\right)$, then, the temperature of the disk, for $R \gg R_{\text{in}}$ is $T_{\text{disk}} = \left(\frac{3GM\dot{M}}{8\pi\sigma_{SB}R_{\text{in}}^3}\right)^{1/4}$ (independent on viscosity).



Spectral energy distribution (SED): $\lambda F_\lambda = \frac{\cos i}{d^2}\int_{R_{\rm in}}^{R_{\rm out}} 2\pi R\lambda B_\lambda[T_s(R)]dR$, where F_λ is the flux, i is the inclination, d is the distance observer-disk (from the solid angle differential). Features: it looks like a "stretched blackbody" spectrum, temperature has a power law of index $n = -4/3$, dependent on temperature profile. Observations, however, show that $T \propto R^{-1/2}$.

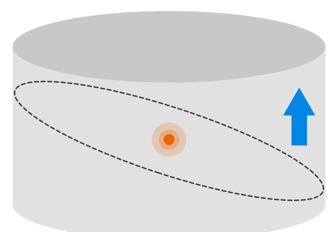
Irradiation: the standard SED does not include stellar irradiation onto the disk. For a disk dominated by radiation (passive disk) $L_{\rm disk} = L_\star \sin\theta$ (disk plane only). Then, $4\pi R^2\sigma T^4 = 4\pi R_\star^2\sigma T_\star^4 \frac{R_\star}{R} \implies T^4 = T_\star^4 R_\star^3/R^3$ $\implies T \propto R^{-3/4}$. This model is not self-consistent, since it predicts that H/R increases with R (the disk flares in the outer regions; detailed models broadly reconcile this behavior with observations).



Model in Chiang & Goldreich 1997 ApJ 490 368: the disk at a given radius has two zones: an upper, hot layer (>blackbody prediction) that re-emits half of radiation and a cold layer that takes the remaining $1/2$ and re-emits in infrared.

Timescales

Dynamical timescale: $t_{\rm dyn} = \Omega^{-1}$ (time of the orbit). For the Keplerian case, $t_{\rm dyn} = \sqrt{\frac{R^3}{GM}} \propto R^{3/2}$.



Vertical timescale: needed to reach hydrostatic balance in the vertical direction. $t_z = \frac{H}{c_s} = \Omega^{-1} = t_{\rm dyn}$. Explanation: the figure shows a Keplerian orbit. If we want the element to stay on top of the imaginary cylinder, we need the pressure to make a force that acts on the timescale of an orbit.

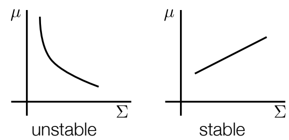
Thermal timescale: when in equilibrium, $t_{\rm heating} = t_{\rm cooling}$. The heating timescale can be approximated by the ratio $t_{\rm th} = \frac{\text{heat content/area}}{\text{accretion power/area}}$. The denominator we know is $\nu\Sigma(R\Omega')^2$. For the numerator, we can use the fact that the internal energy per unit mass is $-P/((\gamma-1)\rho)$. [Explanation: in an adiabatic process, $\Delta U = \Delta W$, but $U = PV/(\gamma-1) \implies U/m = -PV/((\gamma-1)m) = P/((1-\gamma)\rho)$]. Then, the heat content per unit area is $\frac{U}{m}\frac{m}{L^2} = \frac{P}{(1-\gamma)\rho}\Sigma = \frac{c_s^2\Sigma}{\gamma(1-\gamma)}$. This leaves us with the thermal timescale of $t_{\rm th} = \frac{4}{9\gamma(\gamma-1)\alpha\Omega}$, where α is defined via $\nu = \alpha\Omega H^2$, $\mathcal{U}[\alpha] = 1; \alpha < 1$.

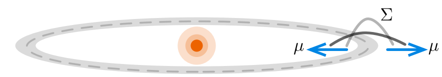
Viscous timescale: $t_\nu = R^2/\nu$.

Ordering of timescales: $t_\nu \gg t_{\rm th} \gg t_z \sim t_{\rm dyn}$.

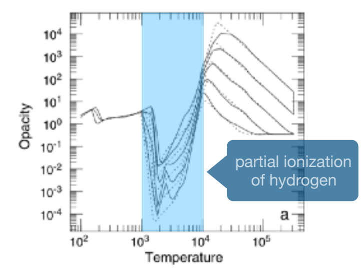
Thermal-viscous instability

$\frac{\partial \Sigma}{\partial t} = \frac{3}{R}\frac{\partial}{\partial R}\left[R^{1/2}\frac{\partial}{\partial R}(\nu \Sigma R^{1/2})\right]$ *is a diffusion equation for* Σ, *but if* $\nu \neq \nu(\Sigma)$. *But if* $\nu = \nu(\Sigma)$, *and defining* $\mu := \mu(\Sigma) := \nu\Sigma$, *we still have a diffusion equation iff* $\frac{\partial \mu}{\partial \Sigma} > 0$. *If it is <0, we have an instability.*

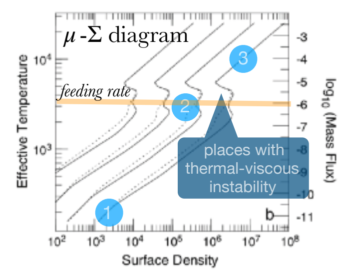




Dimensions of μ: $\mathscr{D}[\mu] = \mathscr{D}[\nu]\mathscr{D}[\Sigma] = \frac{L^2}{T}\cdot\frac{M}{L^2} = \frac{M}{T}$; we can view it as a mass outflow from an annulus.



Interpretation. We look for a function μ such that (since $\nu > 0$ and $\Sigma > 0$) changes sign. For example, if $\nu = C/\Sigma^2$, $\Longrightarrow$ $\mu = C/\Sigma$ $\Longrightarrow$ $\partial\mu/\partial\Sigma = -C/\Sigma^2$. Then, one can say that if the density decreases, the outflow increases! $\Longrightarrow$ unstable; runaway behavior. For the stable configuration, with a decrease in density, the outflow also decreases.



Model by Bell & Lin (1994): the relationship between μ and Σ depends on the thermal structure of the disk. For young stars, Bell & Lin 1994 APJ 307 337 made a model based on detailed opacity laws for circumstellar disks. The system starts at (1), and if it is fed with a bigger rate than the equilibrium one (see orange bar), it will start to move upwards in the curve until the density reaches (2), where the instability is triggered and the flux and temperature jump up to (3), the next upper stable configuration. But then, in (3), the flux is bigger than the feeding rate, so the density decreases until it reaches the instability again and goes to (1). The process repeats itself. The system takes turns in getting to quiescent and outburst phases.

Features for the full disk: a) the model asumes annuli independent of one another. b) The instability is triggered at some radius at which the temperature is high enough to trigger hydrogen ionization. c) Once the instability is triggered, two "fronts" evolve {fast, inwards, "avalanche"} and {slow, outwards, "snowplough"}. d) When the ionization front reaches the innermost parts of the disk, it produces an increase of the optical and infrared flux emitted. e) The disk ends up being characterized by an inner outbursting region and an outer region in the low state. f) After a time $\sim t_\nu$, the instability retreats from the outside in, leaving an inner disk which has been essentially emptied out by the outburst and is ready to slowly be filled up again. This behavior can be seen in dwarf novae.

This instability explains best Herbig Ae/Be stars, which are pre-main-sequence stars with spectral types A, B, < 10 Myr.

FU Orionis objects

FU Orionis objects: small class of protostellar systems undergoing large outbursts, with $\sim 10^3 L_{\mathrm{normal}}$. Very few of these objects have been observed, and only for {F: FU Ori, Va: V1057 Cyg, Vb: V1515 Cyg} we have detailed lightcurves over a long period of time. *Review:* Hartmann & Kenyon 1996 ARA&A

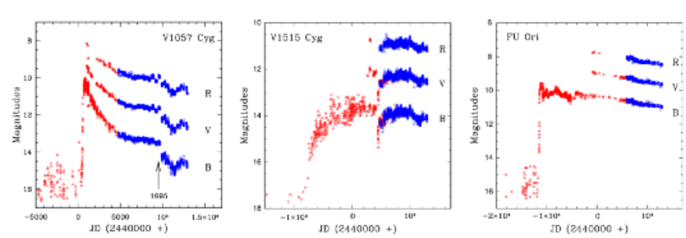


Fig. 9. B, V and R lightcurves of FU Ori, V1057 Cyg and V1515 Cyg. From Clarke et al. (2005).

34, 207.

Features of the lightcurve: a) the lightcurves differ substantially from one another —rise timescale(F, Va) ≈ 1 yr, but rise timescale(Vb) ≈ 10 yr—. b) Va, Vb have shown rapid variability in the high state, while F has not. c) The timescale doesn't allow for repeated outburst observations. d) Duty cycle $\sim 10^4$ years, at least ~ 4 outbursts (from: fraction of FU Orionis objects)

Features of the spectrum: SED consistent with stationary state accretion disk with typical $\dot{M} \sim 10^{-4} M_{\odot}/\mathrm{year}$ for short wavelengths; for mid-infrared, a substantial excess emission is observed. *Observations and explanations*: a) the disk is not in a stationary state, and the stationary state model may not be appropriated. b) Dusty envelope: Kenyon & Hartmann 1991 ApJ 383 664. c) If the outer parts of the disk (beyond ~10 AU) are self-gravitating, more detailed models can reproduce the mid-IR spectrum. d) Disk flaring can be ruled out (degree of flaring needed to reproduce SED is too large).

Interferometric data: a) F: the infrared emission comes from an extended source (consistent with ~few AU disk). b) Other objects: appear more resolved than a simple disk model would predict ($\Longrightarrow$ hotter gas at larger distances, evidence in favor of dust).

Outburst mechanism: the thermal-viscous instability provides outbursts with the correct luminosity and recurrence. It is also able to account for the different rise timescales (caused by the travel time of the instability front): if the instability is triggered in the innermost parts of the disk, the travel time will be determined by the slowly moving outward front (Vb), but if the instability is triggered at some distance, it is the fast inwards "avalanche" the one that determines it (F, Va). A 'hot Jupiter' might be the triggering agent (Clarke & Syer 1996 MNRAS 278 L23). *Defects of the thermal-viscous instability*: the duration is difficult to achieve, it would require a very low viscosity and the duty cycle is one order of magnitude lower than expected. *Alternatives*: a) gravitational instability, b) magneto-rotational instability c) companions? for F, a companion was found at ≈ 200 AU, but the dynamical timescale is not consistent with an outburst.

Viscosity

Where does not *the viscosity come from?* particle collisions (too low ang. mom. transport).

Where the viscosity perhaps comes from? turbulence that arises from MHD instabilities, in particular, the magneto-rotational instability (MRI). It is very probable that something more than simple hydrodynamics must happen in order to have significant ang. mom. transport.

How is viscosity described? α-prescription, which is an adimensional parameter.

Collisional viscosity = (vel. molecules) × (mean free path); where the first factor is $\sim c_s$ and the second one is $\lambda = 1/(n\sigma_{\mathrm{coll}})$ (with σ_{coll} the cross section, n the number density). Then, $\lambda = \dfrac{\mu m_p}{\rho\sigma_{\mathrm{coll}}} = \left(\dfrac{\mu m_p}{\Sigma\sigma_{\mathrm{coll}}}\right)H$, where μ is the average molecular weight ≈ 2. $\therefore \nu = \lambda c_s = \left(\dfrac{\mu m_p H}{\Sigma\sigma_{\mathrm{coll}}}\right)(H\Omega)$

Why collisional viscosity cannot generate enough torque: timescale comparison $\frac{t_\nu}{t_{\rm dyn}} = \frac{R^2\Omega}{\nu}$ = (Reynolds number) $= \frac{\Sigma\sigma_{\rm coll}}{\mu m_p}\left(\frac{H}{R}\right)^{-2}$. With $\sigma_{\rm coll} \approx 10^{-16}{\rm cm}^2$, $\Sigma \approx 0.005 M_\odot/(50\,{\rm AU})^2 \approx 10\,{\rm g/cm}^2$, $H/R \approx 0.1$ $\implies t_\nu/t_{\rm dyn} \approx 10^{11}$. With a dynamical scale of several years, it would take far too long for the mass to be accreted.

Turbulent transport: the Reynolds number is high, and therefore we expect the flow to be highly turbulent.

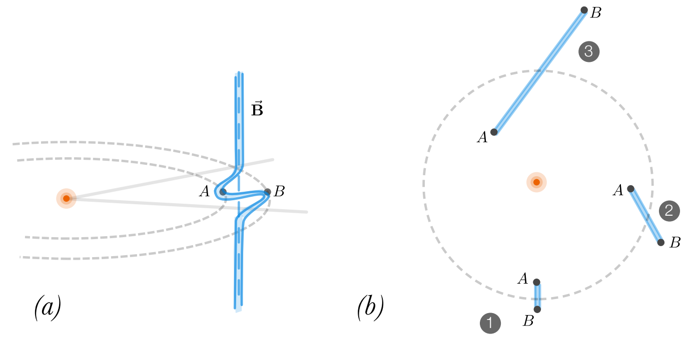
Hydrodynamics, non-viscous vs viscous: (let $\vec{\mathbf{v}}$ be the mean velocity, $\vec{\mathbf{u}}$ the velocity fluctuation, $\langle\rangle$ the average). We write the ang. mom. eq. as $\frac{\partial}{\partial t}(\Sigma R v_\phi) + \frac{1}{R}\frac{\partial}{\partial R}(R v_R \Sigma R v_\phi) = -\sum_i \frac{1}{R}\frac{\partial}{\partial R}\left(R^2\Sigma\langle u_R^{(i)} u_\phi^{(i)}\rangle\right)$. Then, we have the "Reynolds" stress tensor, $T_{R\phi}^{\rm Re} = -\Sigma\left\langle u_R^{\rm Re} u_\phi^{\rm Re}\right\rangle$. In the case of a magnetized disk, the magnetic field $\vec{\mathbf{B}}$ provides another source of transport $\implies$ "Maxwell" stress, $T_{R\phi}^{\rm M} = \Sigma\left\langle u_{A,R} u_{A,\phi}\right\rangle$ where $\vec{\mathbf{u}}_A = \vec{\mathbf{B}}/\sqrt{4\pi\rho}$ G is the Alfvén velocity. If the disk is self-gravitating, the gravitational field provides also another source of transport: $T_{R\phi}^{\rm g} = -\Sigma\left\langle u_R^{\rm g} u_\phi^{\rm g}\right\rangle$, where $\vec{\mathbf{u}}^{\rm g} = \vec{\mathbf{g}}/\sqrt{4\pi G\rho}$. (Ref: Lynden-Bell & Kalnajs 1972 MNRAS 157 1).

Turbulent 'viscosity': the stress provided by turbulent fluctuations behaves like a viscosity, but with a difference: viscosity is a dissipative process (the cause of luminosity in actively accreting disks), while turbulence takes energy from the mean flow and gives it to the fluctuations without real dissipation.

α-prescription and turbulence: the magnitude of the viscosity can be constrained by using the α-prescription. $\mathscr{D}[T_{R\phi}] = \int \mathscr{D}[P]dz = \mathscr{D}[\Sigma]\mathscr{D}[\nu^2]$, and then, we can assume a proportionality to the vertically integrated pressure, $T_{R\phi} = \frac{d\ln\Omega}{d\ln R}\alpha\Sigma c_s^2$, where the derivative term is $\sim -3/2$ for a quasi-Keplerian disk. The relationship with normal viscosity is $\nu = \alpha c_s H$. It is just a change of parameter $\nu \to \alpha$, but a constraint can be made, since we can say that for turbulent viscosity, $\nu \sim \hat{v}l$, with $\hat{v}$ the turbulent velocity ($\lesssim c_s$ or there would be dissipation by shocks) and l the typical size of the largest eddies (maximum: H) which yields $\alpha < 1$, although from observations, $10^{-4} \lesssim \alpha \lesssim 10^{-1}$.

Hydrodynamic instabilities: the Reynolds number is high, but it does not automatically imply that the disk is turbulent, unless there is an energy source to keep the turbulence from decaying. Purely hydrodynamical disks are linearly stable, although they might be non-linearly unstable (debate in progress, Lodato 2008 §8.5).

Epicyclic frequency: imagine a perturbation on a Keplerian orbit of radius R_0 and angular velocity $\Omega_0 = \Omega(R_0)$. Now consider a small displacement from the Keplerian orbit $\vec{\xi}$ such that in the co-rotating frame of reference we can write it as $\vec{\xi} = x\hat{\mathbf{r}} + y\hat{\theta}$ (so x is a small change in radial position). We need the Coriolis force $-2\vec{\mathbf{\Omega}}_0 \times \dot{\vec{\xi}}$ and the variation of the centrifugal force $R[\Omega_0^2 - \Omega^2(R_0 + x)] \approx -xR\frac{d\Omega^2}{dR}$. The equations of motion for the displacement are: $\{\ddot{x} - 2\Omega_0\dot{y} = -xR\frac{d\Omega^2}{dR} + f_x;\quad \ddot{y} + 2\Omega_0\dot{x} = f_y\}$, where f_x, f_y are external forces per unit mass. We consider those forces to be zero for the moment. Those coupled equations can be solved by an Ansatz of the form $e^{i\omega t}$, that yields the condition $\omega^2 = 4\Omega_0^2 + R\frac{d\Omega^2}{dR} := \kappa^2$, which is called *epicyclic frequency*. Instability occurs when $\kappa^2 < 0$ (Rayleigh criterion for instability; Keplerian orbits satisfy the criterion and are stable). Note: we can also deduce this relation with the method described in "Notes on plasma astrophysics"; it is done in Chiuderi & Velli 2015.



Magneto-rotational instability (MRI): (we use the Lagrangian displacement, see "Notes on plasma astrophysics") If we have a magnetic field with lines perpendicular to the disk, like in the figure (a), that magnetic field can suffer from perturbations. In (b), we see that, from above, the magnetic field line is stretched in a way that the change is $\delta\vec{\mathbf{B}} = ikB\vec{\xi}$. This leads to a magnetic tension force of $\frac{ikB}{4\pi\rho}\delta\vec{\mathbf{B}} = -(\vec{\mathbf{k}}\cdot\vec{\mathbf{v}}_A)^2\vec{\xi}$, where $\vec{\mathbf{v}}_A$ is the Alfvén velocity. The magnetic tension behaves like a spring, trying to slow down A, but in the process, gravity acts pulling A further apart towards the center, leading to a runaway behavior. Eventually, the line gets so stretched that magnetic reconnection happens and energy is released, leading to accretion. To see this behavior, we put the magnetic tension as our external force in the equations that we used to derive the epicyclic frequency. Solving again for ω, and using κ as the epicyclic frequency, we have the condition $\omega^4 - \omega^2\left[\kappa^2 + 2(\vec{\mathbf{k}}\cdot\vec{\mathbf{v}}_A)^2\right] + (\vec{\mathbf{k}}\cdot\vec{\mathbf{v}}_A)^2\left[(\vec{\mathbf{k}}\cdot\vec{\mathbf{v}}_A)^2 + \frac{d\Omega^2}{d\ln R}\right] = 0$. The disk becomes unstable when $\omega^2 < 0$, but in the limit of $\vec{\mathbf{v}}_A \to 0$ we do not recover the Rayleigh criterion, but rather $d\Omega^2/d(\ln R) < 0$, which is satisfied by most astrophysically relevant disks. MRI is valid for small magnetic fields. If $v_A > c_s$, the magnetic field has a stabilizing effect.

Gravitational instability: at first glance, only very massive disks are supposed to be self-gravitating, but if the disk is thin, a relatively low mass disks can display some effects of self-gravity. This should be common specially for the earliest stages of star formation where the mass balance of the protostar-disk system is more in favor of the disk (as accretion proceeds, the mass of the disk gets smaller). Self-gravitation can also lead to disk fragmentation. Taking into account disk self-gravitation, we have for small perturbations the dispersion relation $\omega^2 = c_s^2k^2 - 2\pi G\Sigma|k| + \kappa^2$. The first term is the stabilizing effect of pressure, the second one, the destabilizing effect of self gravity and the last term is the stabilizing effect of rotation. $\omega^2 < 0$ iff the dimensionless parameter $Q := \frac{c_s\kappa}{\pi G\Sigma} < 1$. Q was proposed by Toomre 1964 ApJ 139 1217. In the case of $M_{\text{disk}} \ll M$, $Q \approx \frac{c_s\Omega}{\pi G\Sigma} = \frac{c_s}{\Omega R}\frac{GM}{\pi G\Sigma R^2} \approx \frac{H}{R}\frac{M}{M_{\text{disk}}(R)}$, and so, for $Q \sim 1 \implies M_{\text{disk}}/M \sim H/R$.

References

- Lodato 2008 NewAR 52 21 ("Classical disc physics")

- Chiuderi & Velli 2015 *Basics of Plasma Astrophysics, Springer.*

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