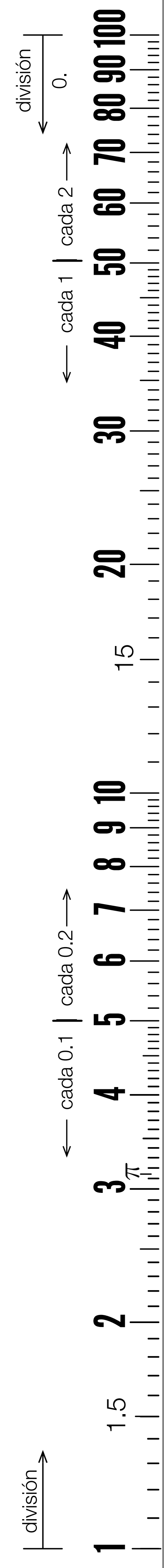
# logarithms
and slide rules

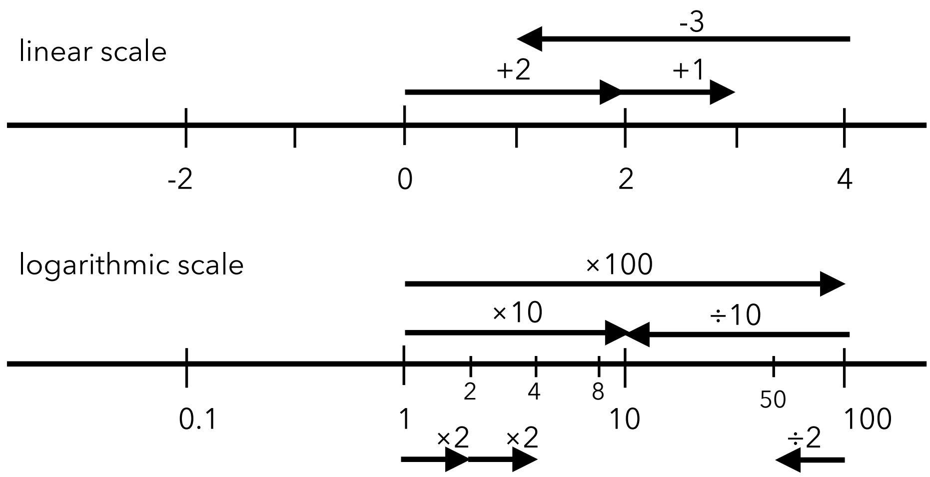



## Basics
A 2-significant figure slide rule works by the property of exponents

$$10^a 10^b = 10^{a+b}$$

A logarithm base 10 is the operation that gives the exponent of a number $x$: if $x = 10^a$, $a = \log x$. The property of exponents in terms of logarithms is $\log a + \log b = \log ab$, which means that a multiplication can be done by addition.

A slide rule has a logarithmic scale. In a linear scale, two numbers can be added or subtracted by moving a certain distance in the scale. In a logarithmic scale, however, moving distances (adding/subtracting) multiplies and divides numbers.




## Algorithm to calculate logarithms
The following algorithm allows an approximation of a logarithm, in principle to an arbitrary number of significant figures. We will consider only two significant figures.

First, we discuss the integer part of a logarithm. We can find it using scientific notation (coefficient $\times 10^{\text{exponent}}$), by reading the exponent. If we have, for example, log 21, we can find the integer part by writing 21 = $2.1\times10^1$, so it's "1.something" (21 it's between $10^1$ and $10^2$). As another example, log 5 is "0.something", since 5 = $5\times10^0$.

The decimal part of the logarithm can be found by raising the coefficient to the power ten, so its logarithm is multiplied by ten: log ($5^{10}$) = 10 log 5, which means that log ($5^{10}$) is between 0 and 10. This way, if we write $5^{10}$ in scientific notation, the exponent is the first digit of the decimal part. We now see some examples with numbers between 0 and 10.

**log 2**

we write $\log(2^{10}) = 10 \log 2$

$2^{10} = \underbrace{2 \times 2 \times 2 \times 2}_{1.6 \cdot 10} \times \underbrace{2 \times 2 \times 2 \times 2}_{1.6 \cdot 10} \times \underbrace{2 \times 2}_{4}$ (A)

$1.6^2 \times 4 \times 10^2$

$2.6 \times 4 \times 10^2$

$1.0 \times 10^{\mathbf{3}}$

(B)

$$\begin{array}{r} 1.6 \times 1.6 \\ \hline 96 \\ 16\phantom{0} \\ \hline 2.56 \end{array}$$

(C)

$$\begin{array}{r} 2.6 \times 4 \\ \hline 10.4 \end{array}$$

the first digit is 3. Then, we say log 2 = 0.3... Reminder: $1.0 \approx 1$

(D) $1^{10} = 1 \times 10^0$

the second digit is 0. Then, we say log 2 = 0.30

Rule: first, determine the integer part by putting the number in scientific notation and reading the exponent. After that, we work with the coefficient as our number.

A. Write the number to the power ten. To facilitate things, group the products so each of them gives a number greater than ten. Write each partial product in scientific notation.

B. Do the products corresponding to the coefficient, rounding the results to two significant figures.

C. If the result of any product is greater than 10, increase by 1 the exponent. When the result is complete, the exponent is the first digit of the logarithm, and the coefficient is the reminder.

D. Repeat the process (A-C) with the reminder, in order to get the second digit of the logarithm, but keep only one significant figure now. When the second digit is obtained, look at the reminder (of the reminder): if it is greater than five, increase the second digit by one (we do this to properly round the logarithm to two significant figures; if this is not done, the result is simply truncated)

## log 3

we write $\log(3^{10}) = 10 \log 3$

$$3^{10} = \underbrace{3 \times 3 \times 3}_{2.7 \cdot 10} \times \underbrace{3 \times 3 \times 3}_{2.7 \cdot 10} \times \underbrace{3 \times 3 \times 3}_{2.7 \cdot 10} \times \underbrace{3}_{3}$$

$$2.7^3 \times 3 \times 10^3$$
$$2.0 \times 3 \times 10^4$$
$$6.0 \times 10^{\mathbf{4}}$$

$2.7 \times 2.7 = 7.29 \approx 7.3$

$7.3 \times 2.7 = 19.71 \approx 2.0 \times 10^1$

$2.0 \times 3 = 6.0$

the first digit is 4. Then, we say log 3 = 0.4... Reminder: $6.0 \approx 6$

$$6^{10} = \underbrace{6 \times 6}_{3.6 \cdot 10} \times \underbrace{6 \times 6}_{3.6 \cdot 10} \times \underbrace{6 \times 6}_{3.6 \cdot 10} \times \underbrace{6 \times 6}_{3.6 \cdot 10} \times \underbrace{6 \times 6}_{3.6 \cdot 10}$$

$$3.6^5 \times 10^5$$
$$1.3 \times 3.6^3 \times 10^6$$
$$1.7 \times 3.6 \times 10^7$$
$$6.1 \times 10^{\mathbf{7}}$$

$3.6 \times 3.6 = 12.96 \approx 1.3 \times 10^1$

$1.3 \times 3.6 = 4.68 \approx 4.7$

$4.7 \times 3.6 = 16.92 \approx 1.7 \times 10^1$

$1.7 \times 3.6 = 6.12 \approx 6.1$

the second digit is 7, but the reminder is $6 > 5$, so, the second digit gets rounded to 8. Then, we say log 3 = 0.48.

## log 7

we write $\log(7^{10}) = 10 \log 7$

$$7^{10} = \underbrace{7 \times 7}_{4.9 \cdot 10} \times \underbrace{7 \times 7}_{4.9 \cdot 10} \times \underbrace{7 \times 7}_{4.9 \cdot 10} \times \underbrace{7 \times 7}_{4.9 \cdot 10} \times \underbrace{7 \times 7}_{4.9 \cdot 10}$$

$$4.9^5 \times 10^5$$
$$2.4 \times 4.9^3 \times 10^6$$
$$1.2 \times 4.9^2 \times 10^7$$
$$2.9 \times 10^{\mathbf{8}}$$

$4.9 \times 4.9 = 24.01 \approx 2.4 \times 10$

$2.4 \times 4.9 = 11.76 \approx 1.2 \times 10$

$1.2 \times 4.9 = 5.88 \approx 5.9$

$5.9 \times 4.9 = 28.91 \approx 2.9 \times 10$

the first digit is 8. Then, we say log 7 = 0.8... Reminder: $2.9 \approx 3$

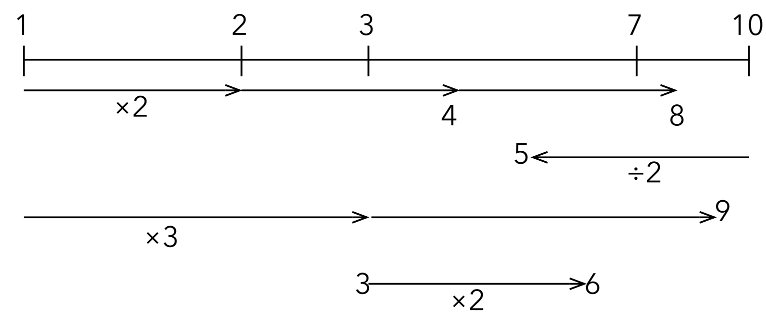
The process has already been done for 3: the second digit is 4 with a reminder of 6 (rounding necessary, second digit becomes 5). Finally, we say log 7 = 0.85.

# Theorems

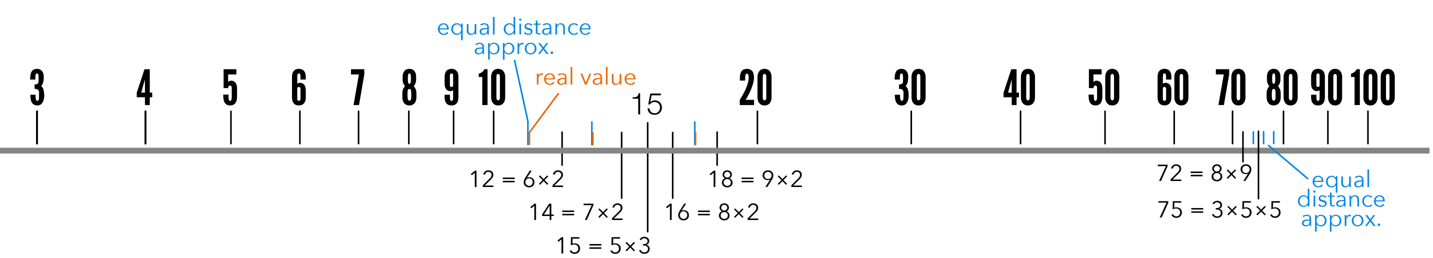
**Fundamental theorem of arithmetic:** Every integer greater than one can be written as a product of prime numbers.

**Result from calculus:** The graph of logarithm is smooth, that means, is locally linear for a small interval.

# How to build a logarithmic scale

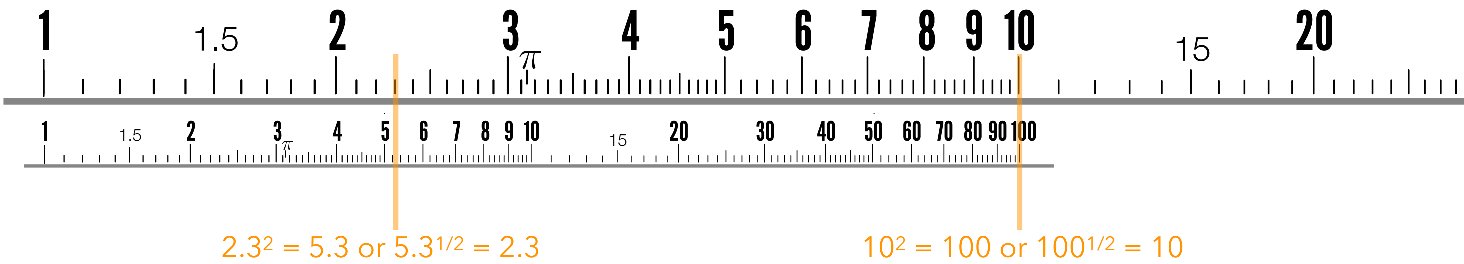



We need first a segment with extremes at the numbers 1 and 10. Then, to construct any number, by the fundamental theorem of arithmetic, we need the logarithms of the prime numbers between 1 and 10, that is, 2, 3, 5 and 7. We first mark 2 (a segment with length log 2) and construct with it the numbers 4 and 8. Now, we recognize that because our base is 10, we can construct the number 5 by dividing 10/2, so we don't actually need log 5 in this special case. Then, we mark 3 and construct with it the number 9, and, together with 2, the number 6. Finally, we put the number 7.

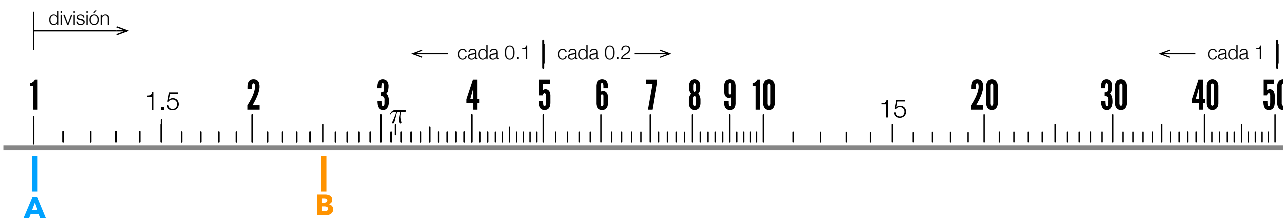



To build the inner divisions of the logarithmic scale, we put two single-digit scales one after another. Now, we can multiply, for example, 3×5 to get 15, 8×2 to get 16, etc. There will be numbers that cannot be reached (11, 23, etc.), because they are primes. However, the marks for these numbers can be more or less safely put in the middle of the two nearest marks. The reason for this is the result from calculus: in small intervals, the log function behaves linearly. In order for this to work, the error must not be greater than the width of the mark.

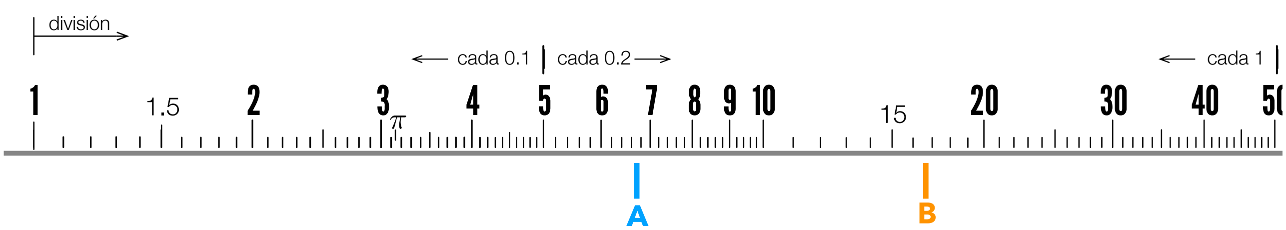
If the whole logarithmic scale is scaled by a factor of two and compared with the unscaled version, one can calculate squares and square roots.



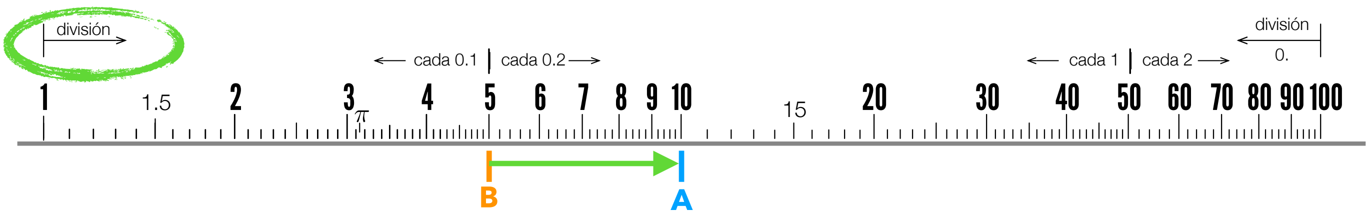

## How to use the slide rule



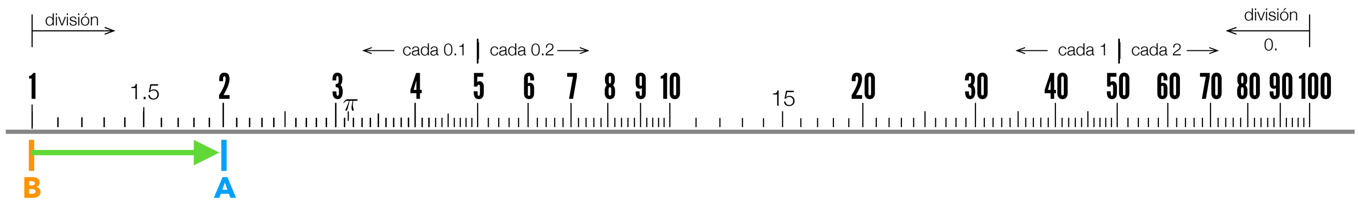

To multiply, we put the log scale over a piece of paper. We make two marks, A and B. A must be in 1 and B is in the first factor we want to multiply. For example, if we want to calculate 2.5×6.7, we put A in 1 and B in 2.5.



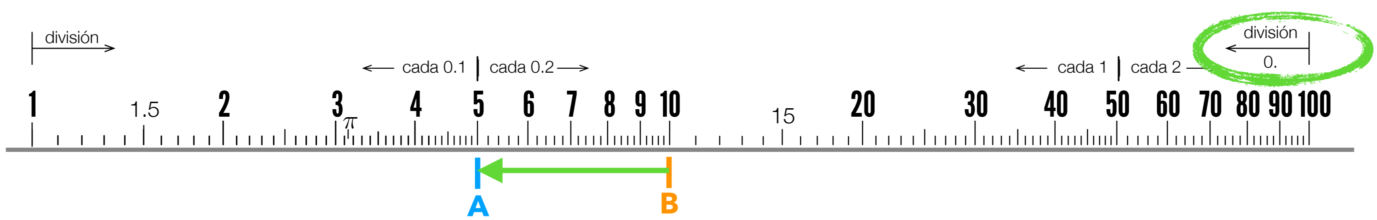

Now, we slide the paper we did the marks in so that A is in the second factor, in this case, in 6.7. B now reads the answer with two significant figures, in this case, 17 (the exact result is 16.75). For two numbers which are not in the interval $0 < x < 10$, we first put the factors in scientific notation. For example, $250 \times 0.0067 = 2.5 \cdot 10^2 \times 6.7 \cdot 10^{-3} = 2.5 \times 6.7 \times 10^{-1}$.

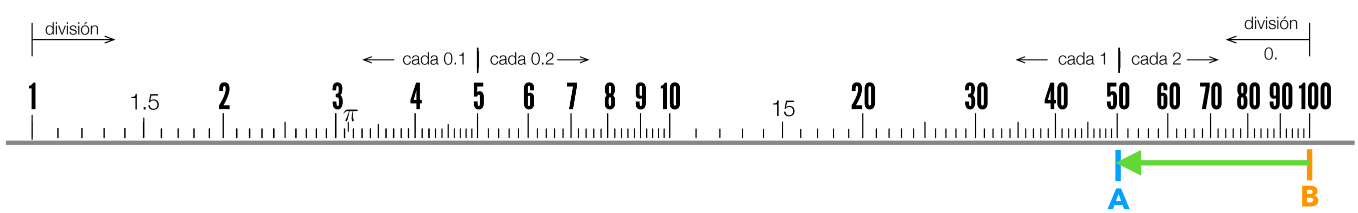



For division, for example 10÷5, we first put A in the dividend (10) and B in the divisor (5). Then, we make an arrow from B to A and locate the division arrow that points in the same direction.




We move the paper so the arrow and the division arrow get aligned (B gets aligned with 1 and A shows the quotient, in this case, 2). So, 10÷5 = 2.




If we do 5÷10, the arrow points to the left, so we align it to the right edge of the scale. In this case, we need to add "0." before reading the answer, so 5÷10 = 0.50.