Gravitational redshift derivation

For people without a GR background

Space-time is described by the variables $x^0 = ct, x^1, x^2, x^3$ (e.g., $= x, y, z$ in Cartesian coordinates). In Euclidian space, we can define the distance between two points by using the Pythagorean theorem: $ds^2_{\text{Eucl}} = dx^2 + dy^2 + dz^2$. In special relativity, however, we know that this distance is not measured the same by an observer moving with respect to us, due to Lorentz contraction. The closest we can define to an invariant "distance" is called *spacetime interval*: $ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$ (this can be shown to be invariant for any observer by using the Lorentz transformations, differentiating and substituting). The meaning of the spacetime interval can be seen by placing ourselves in a frame of reference of a moving particle. In that frame of reference, the particle appears to be static ($dx = dy = dz = 0$), and the measured time is the proper time $d\tau$, so that $ds^2 = -c^2d\tau^2$. Since ds^2 is an invariant, we can always write that relation in any frame.

In general relativity, the spacetime interval also shows the curvature of the spacetime, which is caused by the presence of matter and energy. In order to start the derivation, we are interested in knowing the spacetime interval (sometimes called "the metric") outside of a simple spherical object of mass M. The full solution is called the Schwarzschild solution. As a first order approximation (enough for this derivation), a comparison with Newtonian gravity yields

$$ds^2 \approx -\left(1 + \frac{2\Phi}{c^2}\right)c^2dt^2 + dx^2 + dy^2 + dz^2$$

(this comparison can be done by studying the motion of a particle under a weak potential in classical mechanics, with a Lagrangian, and relativity, by saying that the trajectory of a particle has to minimize the proper time [variational calculus]). For a spherical object of mass M the potential is $\Phi = -GM/r$.

In this metric, now, t is defined as measured at infinity. We see this by considering a static observer ($dx = dy = dz = 0$) at $r \to \infty$ and writing $ds^2 = -c^2d\tau^2 = -\left(1 - \frac{2G\cancel{M}}{\cancel{r}c^2}\right)c^2dt^2 \implies dt_\infty = d\tau_\infty$. This means that the relation between the proper time measured by an observer at a radius r and the time measured at infinity is $d\tau = \sqrt{1 - \frac{2GM}{rc^2}}dt$.

Consider the propagation of an electromagnetic wave from the surface of an object of mass M and radius R to infinity. The time difference between two moments in which the wave has the same phase is the period T, and an observer at infinity will measure a different period than an observer at the surface: $T_{\text{surf}} = \sqrt{1 - \frac{2GM}{Rc^2}}T_\infty \implies f_{\text{surf}} = \frac{1}{\sqrt{1 - \frac{2GM}{Rc^2}}}f_\infty$ (f is the frequency of the wave), or, in terms of the wavelength $\lambda = c/f$, $\implies \frac{\lambda_\infty}{\lambda_{\text{surf}}} = \frac{1}{\sqrt{1 - \frac{2GM}{Rc^2}}} := 1 + z$.