

Problem: Find all the spherical harmonics for $\ell=1$

Solution:

From the definition, we know that $-\ell \leq m \leq \ell$.

This means we need

$$\ell=1$$

$$m=-1, 0, 1$$

Then,

$$Y_{1,1}(\theta, \varphi) \propto P_1^1(\cos\theta) e^{i\varphi}$$

(normalization:

$$\sqrt{\frac{2\ell+1}{4\pi} \cdot \frac{(\ell-m)!}{(\ell+m)!}} = \sqrt{\frac{3}{4\pi} \cdot \frac{0!}{2!}} = \sqrt{\frac{3}{8\pi}}$$

$$= -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\varphi}$$

$$Y_{1,0}(\theta, \varphi) \propto P_1^0(\cos\theta) e^0$$

(normalization = $\sqrt{\frac{3}{4\pi} \cdot \frac{1!}{1!}} = \sqrt{\frac{3}{4\pi}}$

$$= \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$Y_{1,-1}(\theta, \varphi) \propto P_1^{-1}(\cos\theta) e^{-i\varphi}$$

We know that

$$Y_{\ell-m} = (-1)^m \overline{Y_{\ell m}}$$

$$= -\overline{Y_{1,1}} = \sqrt{\frac{3}{8\pi}} \sin\theta e^{-i\varphi}$$

comes from properties of assoc. Legendre and can be shown by Rodrigues's formula

Problem: Write the functions $f(\theta) = \cos\theta$, $g(\theta) = \sin\theta$

in terms of the spherical harmonics.

► $f(\theta)$ is the Legendre polynomial $P_1(\cos\theta) = \cos\theta$.

We try by inspection:

• with $\ell=0$, $m=0$

$$Y_{0,0}(\theta, \varphi) \propto P_0(\cos\theta) e^0 = 1 \quad \times \quad \text{for } f(\theta)$$

• with $l=1$, $m=-1, 0, 1$

$$Y_{1,0}(\theta, \varphi) \propto \cos \theta \quad \checkmark$$

for $f(\theta)$

$$Y_{1,1}(\theta, \varphi) \propto \sin \theta \quad \times$$

$$\text{So, } f(\theta) = \underbrace{\left(\sqrt{\frac{4\pi}{3}}\right)}_{\text{coeff}} \sqrt{\frac{3}{4\pi}} \cos \theta = \sqrt{\frac{4\pi}{3}} Y_{1,0}(\theta, \varphi)$$

- For higher order l sph. harmonics, and $m=0$, we get a higher order Legendre polynomial $\Rightarrow$ by orthogonality, $l > 1$ won't work.
- For higher order l and $m \neq 0$, none will work because of the φ dependency ($e^{im\varphi}$)

✦ for $g(\theta)$, we see that

$$Y_{1,0}(\theta, \varphi) \propto \cos \theta \quad \times$$

$$Y_{1,1}(\theta, \varphi) \propto \sin \theta e^{i\varphi} \quad \times$$

$$Y_{1,-1}(\theta, \varphi) \propto \sin \theta e^{-i\varphi} \quad \times$$

Then this means we must expand in this case as

$$g(\theta) = \sum_{l=0}^{\infty} \sum_{m=-l}^l g_{lm} Y_{lm}(\theta, \varphi)$$

but we know that none of the $m \neq 0$ will work because there is no φ dependency. This means

$$g(\theta) = \sum_{l=0}^{\infty} g_{l0} Y_{l0}(\theta, \varphi)$$

which is equivalent to the problem of expanding $\sin \theta$ as a series of Leg. polyn. eval. on $\cos \theta$, that is

$$\sin \theta = \sum_{l=0}^{\infty} \tilde{g}_l P_l(\cos \theta)$$

Note how different the solution becomes by just rotating the coordinate θ . For solving problems, one should exploit the symmetries so that the solution is as simple as possible.

Problem: Show that

$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x)$$

Part I: first we show that

$$\frac{d^{l-m}}{dx^{l-m}} (x^2-1)^l = \frac{(l-m)!}{(l+m)!} (x^2-1)^m \frac{d^{l+m}}{dx^{l+m}} (x^2-1)^l$$

First we write

$$(x^2-1)^l = (x-1)^l (x+1)^l$$

Now we use Leibniz rule:

$$\begin{aligned} \textcircled{A}: \frac{d^{l-m}}{dx^{l-m}} (x-1)^l (x+1)^l &= \sum_{k=0}^{l-m} \frac{(l-m)!}{k! (l-m-k)!} \frac{d^{l-m-k}}{dx^{l-m-k}} (x-1)^l \frac{d^k}{dx^k} (x+1)^l \\ &= \sum_{k=0}^{l-m} \frac{(l-m)!}{k! (l-m-k)!} \frac{l! (x-1)^{l+k}}{(l-l+m+k)!} \frac{l! (x+1)^{l-k}}{(l-k)!} \end{aligned}$$

Leibniz diff. rule:

$$\begin{aligned} \frac{d^n}{dx^n} (fg) &= \sum_{k=0}^n \binom{n}{k} \frac{d^{n-k} f}{dx^{n-k}} \frac{d^k g}{dx^k} \\ \binom{n}{k} &= \frac{n(n-1)(n-2)\dots(n-k+1)}{k!} \\ &= \frac{n!}{k! (n-k)!} \end{aligned}$$

$$\begin{aligned} \frac{d^p}{dx^p} (x^n) &= n(n-1)\dots(n-p+1) x^{n-p} \\ &= \frac{n!}{(n-p)!} x^{n-p} \end{aligned}$$

$$\begin{aligned} \textcircled{B}: \frac{d^{l+m}}{dx^{l+m}} (x-1)^l (x+1)^l &= \sum_{k=0}^{l+m} \frac{(l+m)!}{k! (l+m-k)!} \frac{d^{l+m-k}}{dx^{l+m-k}} (x-1)^l \frac{d^k}{dx^k} (x+1)^l \\ &\quad \begin{array}{cc} \text{but } \frac{l+m-k \leq l}{\Rightarrow k \geq m} & \text{but } k \leq l \end{array} \end{aligned}$$

$$= \sum_{k=m}^l \frac{(l+m)!}{k! (l+m-k)!} \frac{d^{l+m-k}}{dx^{l+m-k}} (x-1)^l \frac{d^k}{dx^k} (x+1)^l$$

$$\text{let } j = k-m. \text{ When } k=m \Rightarrow j=0 \\ \Rightarrow k=j+m \quad k=l \Rightarrow j=l-m$$

$$= \sum_{j=0}^{l-m} \frac{(l+m)!}{(j+m)! (l+m-j-m)!} \frac{d^{l-j}}{dx^{l-j}} (x-1)^l \frac{d^{j+m}}{dx^{j+m}} (x+1)^l$$

$$= \sum_{k=0}^{l-m} \frac{(l+m)!}{(k+m)! (l-k)!} \frac{l!}{(l-l+k)!} (x-1)^k \frac{l!}{(l-k-m)!} (x+1)^{l-k-m}$$

Comparing,

$$\textcircled{A} \quad \sum_{k=0}^{l-m} \frac{(l-m)!}{k!(l-m-k)!} \frac{l! (x-1)^{m+k}}{(l-k+m+k)!} \frac{l! (x+1)^{l-k}}{(l-k)!}$$

$$\textcircled{B} \quad \sum_{k=0}^{l-m} \frac{(l+m)!}{(k+m)!(l-k)!} \frac{l!}{(l-k-k)!} (x-1)^k \frac{l!}{(l-k-m)!} (x+1)^{l-k-m}$$

We see that

$$\textcircled{A} = \frac{(l-m)!}{(l+m)!} (x-1)^m (x+1)^m \textcircled{B}$$

which verifies the expression.

Part II: Now we start with the Rodrigues's formula for $P_l^{-m}(x)$

$$P_l^{-m}(x) = \frac{(-1)^m}{2^l l!} (1-x^2)^{-m/2} \frac{d^{l-m}}{dx^{l-m}} (x^2-1)^l$$

$$= \frac{(-1)^m}{2^l l!} (1-x^2)^{-m/2} \frac{(l-m)!}{(l+m)!} \underbrace{(x^2-1)^m}_{(-1)^m (1-x^2)^m} \frac{d^{l+m}}{dx^{l+m}} (x^2-1)^l$$

$$\Rightarrow P_l^{-m}(x) = \frac{(-1)^m (-1)^m (1-x^2)^{m/2}}{2^l l!} \frac{(l-m)!}{(l+m)!} \frac{d^{l+m}}{dx^{l+m}} (x^2-1)^l$$

$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x)$$