

Problem: show the normalization of the Legendre polynomials.

Solution method I: using the recursion formula.

We have to show that

$$\int_{-1}^1 [P_\ell(x)]^2 dx = \frac{2}{2\ell+1}$$

We start from the recursion formula

$$\ell P_\ell(x) = x P'_\ell(x) - P'_{\ell-1}(x)$$

and substitute

$$\ell \int_{-1}^1 [P_\ell(x)]^2 dx = \int_{-1}^1 x P_\ell(x) P'_\ell(x) dx - \int_{-1}^1 P_\ell(x) P'_{\ell-1}(x) dx$$

this is a polynomial of order $\ell-1$
 $\Rightarrow$ orthogonal with $P_\ell(x)$
 $= 0$

$$\ell \int_{-1}^1 [P_\ell(x)]^2 dx = \frac{x}{2} [P_\ell(x)]^2 \Big|_{-1}^1 - \frac{1}{2} \int_{-1}^1 [P_\ell(x)]^2 dx$$

parts
 $= 1$ at boundaries
 $:= I$

$$\ell I = 1 - \frac{1}{2} I$$

$$\Rightarrow I = \frac{2}{2\ell+1} \quad \checkmark$$

Solution method II: using the generating function.

The generating function is $\frac{1}{\sqrt{1-2t\mu+t^2}} = \sum_{\ell=0}^{\infty} P_\ell(\mu) t^\ell$

We now square it and integrate over μ on both sides in order to form the normalization integral inside of the sum

$$\int_{-1}^1 \frac{1}{1-2t\mu+t^2} d\mu = \sum_{\ell=0}^{\infty} t^{2\ell} \int_{-1}^1 [P_\ell(\mu)]^2 d\mu$$

$$= \frac{1}{t} \ln \frac{1+t}{1-t}$$

Now we expand the left hand side in order to also form a series there. Our goal is to compare coefficients of the expansion in powers of t in the end.,

$$\frac{1}{t} \ln \frac{1+t}{1-t} = 2 + \frac{2}{3}t^2 + \frac{2}{5}t^4 + \dots + \frac{2}{2l+1}t^{2l} + \dots$$

$$= \sum_{l=0}^{\infty} t^{2l} \cdot \frac{2}{2l+1}$$

$$\sum_{l=0}^{\infty} t^{2l} \cdot \frac{2}{2l+1} = \sum_{l=0}^{\infty} t^{2l} \int_{-1}^1 [P_l(\mu)]^2 d\mu$$

$$\Rightarrow \int_{-1}^1 [P_l(\mu)]^2 d\mu = \frac{2}{2l+1}$$

Tables:

$$\frac{1}{2} \ln \left(\frac{1+x}{1-x} \right) = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$$

Problem: compute the expansion of the function

$$f(x) = \sqrt{\frac{1-x}{2}}$$

as a series in terms of the Legendre polynomials.

Solution: the general procedure for computing the coefficients of the series expansion (inner product from the left $\rightarrow$ orthogonality $\rightarrow$ solving for the coefficients) can also be applied to the generating function. We try this calculation way for this exercise:

Multiplying the generating function by $\sqrt{\frac{1-x}{2}}$ and integrating,

$$\underbrace{\int_{-1}^1 \frac{\sqrt{1-x}}{\sqrt{2}} \frac{1}{\sqrt{1-2xt+t^2}} dx}_{\text{rationalize \& substitute to obtain}} = \sum_{n=0}^{\infty} t^n \int_{-1}^1 \sqrt{\frac{1-x}{2}} P_n(x) dx$$

$$= \frac{1}{2t} \left[1+t - \frac{(1-t)^2}{2\sqrt{t}} \ln \frac{1+\sqrt{t}}{1-\sqrt{t}} \right]$$

expanding the l.h.s. as a power series of t ,

$$\dots \Rightarrow \frac{4}{3} - 4 \sum_{n=1}^{\infty} \frac{t^n}{(4n^2-1)(2n+3)} = \sum_{n=0}^{\infty} t^n \int_{-1}^1 \sqrt{\frac{1-x}{2}} P_n(x) dx$$

comparing coefficients,

$$\int_{-1}^1 \sqrt{\frac{1-x}{2}} P_0(x) dx \stackrel{!}{=} \frac{4}{3}$$

$$\int_{-1}^1 \sqrt{\frac{1-x}{2}} P_n(x) dx \stackrel{!}{=} -\frac{4}{(4n^2-1)(2n+3)}$$

$$\Rightarrow \sqrt{\frac{1-x}{2}} = \frac{2}{3} P_0(x) - 2 \sum_{n=1}^{\infty} \frac{P_n(x)}{(2n-1)(2n+3)}$$

Problem: compute the integral

$$\int_0^1 P_l(\mu) d\mu$$

Solution: we use the generating function and a subsequent expansion in powers of t in order to get the integral

$$\int_0^1 \frac{1}{\sqrt{1-2t\mu+t^2}} d\mu = \int_0^1 \sum_{l=0}^{\infty} t^l P_l(\mu) d\mu$$

$$\text{l.h.s.: } \int_0^1 \frac{1}{\sqrt{1-2t\mu+t^2}} d\mu = -\frac{1}{t} \sqrt{1-2t\mu+t^2} \Big|_0^1 = \frac{1}{t} [\sqrt{1+t^2} - (1-t)]$$

expansion as a power series

$$= 1 + \sum_{n=1}^{\infty} \binom{1/2}{n} t^{2n-1}$$

$$\Rightarrow 1 + \sum_{n=1}^{\infty} \binom{1/2}{n} t^{2n-1} = \sum_{l=0}^{\infty} t^l \int_0^1 P_l(\mu) d\mu$$

$$\Rightarrow 1 + \sum_{n=1}^{\infty} \binom{1/2}{n} t^{2n-1} = 1 + \sum_{l=1}^{\infty} t^l \int_0^1 P_l(\mu) d\mu$$

comparing,

$$\int_0^1 P_l(\mu) d\mu = \begin{cases} 1, & l=0 \\ 0, & l \text{ even (but not 0)} \\ \binom{1/2}{n}, & l=2n-1 \text{ (l odd)} \end{cases}$$

■

Tables:

Binomial series:

$$(1+x)^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

$$\frac{1}{t} \left[1 + t + \sum_{k=0}^{\infty} \binom{1/2}{k} t^{2k} \right]$$

first term:

$$\binom{1/2}{0} t^0 = \frac{\Gamma(1/2+1)}{\Gamma(0+1)\Gamma(1/2-0+1)} = \frac{\Gamma(3/2)}{1 \cdot \Gamma(1/2)} = 1$$

$$\Rightarrow \frac{1}{t} \left[t + \sum_{k=1}^{\infty} \binom{1/2}{k} t^{2k} \right]$$

Problem: Obtain a series expansion of the function

$$f(x) = H(x) = \begin{cases} 1, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

in terms of the Legendre polynomials.

Solution:

The coefficients are given by

$$c_l = \frac{\langle f(x), P_l(x) \rangle}{N_l^2} = \frac{2l+1}{2} \int_{-1}^1 H(x) P_l(x) dx = \frac{2l+1}{2} \int_0^1 P_l(x) dx$$

which is the previous exercise, and can be re-written in several ways.

To test the expansion, we plot the first 100 coefficients, that is

$$H_\ell(x) \approx \sum_{\ell=0}^{100} c_\ell P_\ell(x)$$

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.special import gamma, legendre
```

✓ 0.0s

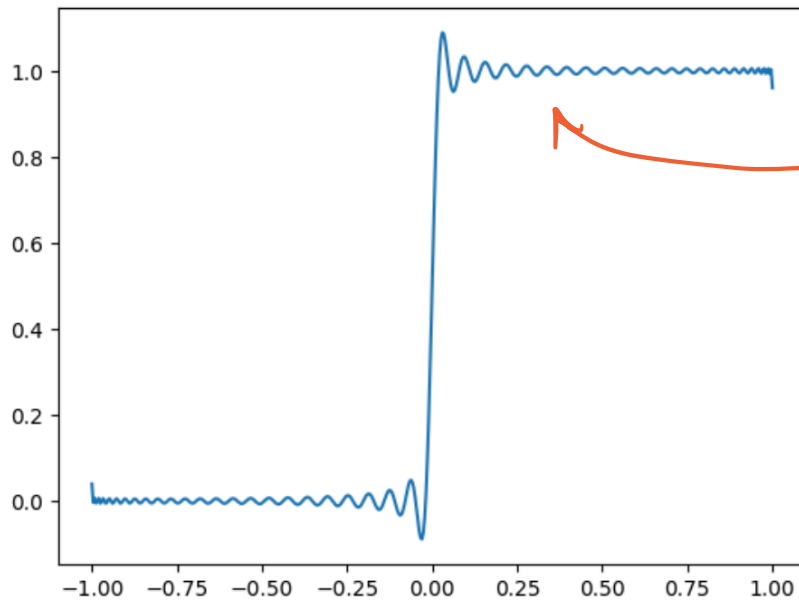
```
def coeff(l):
    norm = (2*l + 1)/2
    if l == 0: return 1 * norm
    elif l % 2 == 0: return 0
    else:
        n = (l + 1)//2
        return norm * gamma(1/2+1)/(gamma(n+1)*gamma(1/2-n+1))
```

✓ 0.0s

```
x = np.linspace(-1,1,500)
order = 100
series_approx = 0
for i in range(order):
    series_approx += coeff(i)*legendre(i)(x)
plt.plot(x, series_approx)
```

✓ 0.0s

[<matplotlib.lines.Line2D at 0x11ab52600>]



Problem: show that

$$\int_{-1}^1 x^m P_n(x) dx = 0 \quad \text{for } m < n.$$

Solution:

Using the Rodrigues' formula,

$$\begin{aligned} & \int_{-1}^1 x^m \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n dx \\ & \text{parts once:} \\ & = \frac{x^m}{2^n n!} \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n \Big|_{-1}^1 - \int_{-1}^1 \frac{d}{dx} (x^m) \frac{1}{2^n n!} \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n dx \\ & \vdots \\ & \text{parts } n \text{ times} \\ & = (-1)^n \int_{-1}^1 \frac{d^n}{dx^n} (x^m) \frac{1}{2^n n!} (x^2-1)^n dx \\ & \text{if } n > m, \text{ this term will be zero. } \quad \square \end{aligned}$$

Problem: Expand the function

$$f(x) = \begin{cases} 0, & -1 \leq x < \alpha \\ 1, & \alpha < x \leq 1 \end{cases} \quad \text{i.e., } H(x-\alpha)$$

as a Legendre series in the interval $[-1, 1]$ **Solution:** We compute the coefficients in a naïve way

$$c_n = \frac{2n+1}{2} \int_{\alpha}^1 P_n(x) dx$$

We solve this integral by substituting a recurrence formula

$$\begin{aligned} c_n &= \frac{2n+1}{2} \int_{\alpha}^1 \frac{P'_{n+1}(x) - P'_{n-1}(x)}{2n+1} dx \quad \text{for } n \geq 1 \\ &= \frac{1}{2} \left[\int_{\alpha}^1 P'_{n+1}(x) dx - \int_{\alpha}^1 P'_{n-1}(x) dx \right] \\ &= \frac{1}{2} [P_{n+1}(x) - P_{n-1}(x)]_{\alpha}^1 \\ & \text{but from the b.c., we know that } P_n(1) = 1, \text{ so,} \\ &= \frac{1}{2} [1 - P_{n+1}(\alpha) - 1 + P_{n-1}(\alpha)] \\ &= \frac{1}{2} [P_{n-1}(\alpha) - P_{n+1}(\alpha)] \end{aligned}$$

$$\therefore f(x) = c_0 P_0(x) + \sum_{n=1}^{\infty} c_n P_n(x) = \frac{1}{2}(1-\alpha) + \sum_{n=1}^{\infty} \frac{1}{2} [P_{n-1}(\alpha) - P_{n+1}(\alpha)] P_n(x)$$

Note: for $n=0$, the recurrence relation is not valid (we would get $P_{-1}(x)$).

We do the integral separately:

$$\begin{aligned} c_0 &= \frac{1}{2} \int_{\alpha}^1 P_0(x) dx = \frac{1}{2} \int_{\alpha}^1 1 dx = \frac{1}{2} [x]_{\alpha}^1 \\ &= \frac{1}{2} (1-\alpha) \end{aligned}$$

Associated Legendre equation

The associated Legendre equation is a generalization of the Legendre equation given by

$$\frac{d}{dx} \left((1-x^2) y'(x) \right) + \left(\ell(\ell+1) - \frac{m^2}{1-x^2} \right) y(x) = 0, \quad m^2 \leq \ell^2$$

for the case $m=0$ we recover the Legendre ODE.

We propose the Ansatz

$$y(x) = (1-x^2)^{m/2} v(x)$$

and after substitution, one can show that this yields

$$(1-x^2) v''(x) - 2(m+1)x v'(x) + [\ell(\ell+1) - m(m+1)] v(x) = 0 \quad (*)$$

If we then differentiate both sides of this equation, we get

$$(1-x^2) [v'(x)]'' - 2(m+2)x [v'(x)]' + (\ell(\ell+1) - (m+1)(m+2)) [v'(x)] = 0$$

We notice that this last equation is the same as equation (*) with the substitutions $m \rightarrow m+1$, $v(x) \rightarrow v'(x)$.

Now we notice that we have

- $y(x) = v(x) = P_\ell(x)$ is a solution for $m=0$ [eq. (*) with $m=0$ is the Legendre ODE]
- $v(x) = P_\ell'(x)$ " " " " $m=1$
 $\Rightarrow y(x) = (1-x^2)^{1/2} P_\ell'(x)$
- $v(x) = P_\ell''(x)$ " " " " $m=2$
 $\Rightarrow y(x) = (1-x^2) P_\ell''(x)$
- $\vdots$

Then we define:

The **Associated Legendre Functions** as $y(x) := P_\ell^m(x) := (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x)$, $0 \leq m \leq \ell$ conventional

and the **Associated Legendre Polynomials** as $v(x) := \mathcal{P}_\ell^m(x) := (-1)^m \frac{d^m}{dx^m} P_\ell(x)$

WARNING: the notation and normalization is different in every book!

Problem: find the Rodrigue's formula for the assoc. Legendre functions.

$$P_\ell^m = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} \left(\frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} (x^2-1)^\ell \right)$$

$$= \frac{(-1)^m (1-x^2)^{m/2}}{2^\ell \ell!} \frac{d^{\ell+m}}{dx^{\ell+m}} (x^2-1)^\ell, \quad \ell \geq 0, |m| \leq \ell, m \in \mathbb{Z}, \ell \in \mathbb{N}_0$$

Problem. Obtain a recurrence relation of the associated Legendre polynomials for the index m .

Solution: starting with the ODE

$$(1-x^2)y'' - 2xy' + \left[l(l+1) - \frac{m^2}{1-x^2}\right]y = 0$$

we introduce the Ansatz $y = (x^2-1)^{m/2}v$ and obtain

$$(1-x^2)v'' - 2(m+1)xv' + (l-m)(l+m+1)v = 0$$

We now set $v = d^m P_l(x)/dx^m$
and multiply by $(x^2-1)^{m/2}$

$$\left[(1-x^2) \frac{d^{m+2}}{dx^{m+2}} P_l(x) - 2(m+1)x \frac{d^{m+1}}{dx^{m+1}} P_l(x) + (l-m)(l+m+1) \frac{d^m}{dx^m} P_l(x) \right] \cdot (x^2-1)^{m/2} = 0$$

$\downarrow$
 $P_l^{m+2}(x)$

$$\Rightarrow P_l^{m+2}(x) + \frac{2(m+1)x}{(x^2-1)^{1/2}} P_l^{m+1}(x) - (l-m)(l+m+1) P_l^m(x) = 0$$

Problem. Compute the normalization of the associated Legendre functions.

Hints: a) A recursion relation satisfied by the assoc. Legendre funct. is

$$(2l+1)\mu P_l^m = (l-m+1)P_{l+1}^m + (l+m)P_{l-1}^m ; \quad P_l^m = P_l^m(\mu)$$

b) It can be shown that

$$P_m^m(\mu) = \frac{(2m)!}{m! 2^m} \sin^m \theta$$

c) The following integral is known:

$$\int_0^\pi \sin^{2m+1} \theta \, d\theta = \frac{2^{2m+1} (m!)^2}{(2m+1)!}$$

Solution: We want to compute the integral

$$\int_{-1}^1 [P_l^m]^2 d\mu \quad , \quad \text{where } \mu = \cos \theta, \quad P_l^m = P_l^m(\mu)$$

We start from the recurrence relation to form this integral.

$$\int_{-1}^1 \left[(2l+1)\mu P_l^m = (l-m+1)P_{l+1}^m + (l+m)P_{l-1}^m \right] \cdot P_{l-1}^m d\mu$$

using orthogonality, $\int_{-1}^1 P_l^m P_{l'}^m d\mu = 0$,

$$\Rightarrow (2l+1) \int_{-1}^1 P_l^m P_{l-1}^m d\mu = (l+m) \int_{-1}^1 [P_{l-1}^m]^2 d\mu$$

$\leftarrow$ use recurrence formula again to substitute

$$\Rightarrow (l-m) \frac{(2l+1)}{(2l-1)} \int_{-1}^1 [P_l^m(\mu)]^2 d\mu = (l+m) \int_{-1}^1 [P_{l-1}^m(\mu)]^2 d\mu$$

Repeating this procedure until $l=m \dots$

$$\dots \rightarrow \int_{-1}^1 [P_l^m(\mu)]^2 d\mu = \frac{2m+1}{2l+1} \frac{(l+m)!}{(2m)!(l-m)!} \int_{-1}^1 [P_m^m(\mu)]^2 d\mu$$

Using the other two hints, we obtain

$$\int_{-1}^1 [P_l^m(\mu)]^2 d\mu = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!}$$

Tema/Topic: Legendre functions of the second kind

Legendre Functions of the second kind

The Legendre ODE is a second order ODE. This means that there must be a second kind of solution, linearly independent of the Legendre polynomials.

Review: Given a differential equation of the form

$$y'' + F(x)y' + G(x)y = 0$$

the Wronskian is, by definition, $W = y_1 y_2' - y_1' y_2$

and differentiating the Wronskian one can show

$$W' = -F(x)W$$

Now, we separate variables and integrate

$$\frac{dW}{W} = -F(x) dx$$

$$\Rightarrow \ln \frac{W(x)}{W(0)} = - \int_0^x F(x_1) dx_1$$

But

$$W(x) = y_1 y_2' - y_1' y_2 = y_1^2 \frac{d}{dx} \left(\frac{y_2}{y_1} \right)$$

$$\Rightarrow \frac{d}{dx} \left(\frac{y_2}{y_1} \right) = W(x) \frac{\exp \left[- \int_0^x F(x_1) dx_1 \right]}{y_1^2}$$

$$\Rightarrow y_2(x) = y_1(x) \int_0^x \frac{\exp \left[- \int_0^x F(x_1) dx_1 \right]}{[y_1(x_2)]^2} dx_2$$

(arbitrarily setting $W(0)=1$).

For the Legendre ODE,

$$y'' - \frac{2x}{1-x^2} y' - \frac{\ell(\ell+1)}{1-x^2} y = 0$$

the second solutions $Q_\ell(x)$ are called **Legendre functions of the second kind**.

$$Q_\ell(x) = P_\ell(x) \int_0^x \frac{\exp \left[\int_0^x \frac{2x}{1-x^2} dx \right]}{[P_\ell(x)]^2}$$

For $\ell=0$

$$Q_0(x) = P_0(x) \int_0^x \frac{1}{1-x^2} \frac{1}{[P_0(x)]^2} dx = \int_0^x \frac{1}{1-x^2} dx = \frac{1}{2} \int \left[\frac{1}{1+x} + \frac{1}{1-x} \right] dx$$

$$\Rightarrow Q_0(x) = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right)$$

For $l=1$:

$$Q_1(x) = x \int \frac{dx}{(1-x^2)x^2} = \frac{x}{2} \ln \left(\frac{1+x}{1-x} \right) - 1$$

For the rest of the l , we can use the same recurrence formulae as for the Legendre polynomials, for example

$$(l+1)Q_{l+1}(x) - (2l+1)xQ_l(x) + lQ_{l-1}(x) = 0.$$

Note: there are also associated Legendre functions of the second kind for the associated Legendre ODE.

Bessel functions**Bessel's equation**

When we solve the ODE

$$x^2 y'' + x y' + (x^2 - p^2) y = 0$$

by the series solution method, we find an explicit expression for the Bessel functions of the 1st kind of order p :

$$J_p(x) \equiv J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1) \Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n+p}$$

for integer $p > 0$. For integer $p < 0$, the $\Gamma(n-p+1)$ function diverges if $n \leq p-1$

Then, the first terms of the series are going to be zero ($1/\infty \rightarrow 0$)

until the series starts as in the positive case. One can show that

$$J_{-p}(x) = (-1)^p J_p(x)$$

so that the solution with positive and negative integer p are not linearly independent. We need to find another solution

Bessel functions of the second kind

For integer p , the function

$$Y_p(x) = \lim_{\nu \rightarrow p} \frac{\cos(\pi\nu) J_\nu(x) - J_{-\nu}(x)}{\sin(\pi\nu)} \quad (\text{also } := N_p(x))$$

provides a linearly independent solution to the Bessel ODE. (Neumann or Weber function)

In summary, the full solution is

$$y(x) = A J_p(x) + B Y_p(x), \quad p \in \mathbb{Z}$$

Note that if p is not an integer, we can use the negative p as a linearly independent solution and in that case, the general solution is

$$y(x) = A J_p(x) + B J_{-p}(x), \quad p \text{ non-integer}$$

Bessel ODE as a Sturm-Liouville problem

The differential operator of the Bessel ODE can be written as

$$x^2 y'' + x y' + (x^2 - p^2) y = 0 \quad \Rightarrow \quad x \frac{d}{dx} \left(x \frac{d}{dx} \right) y + (x^2 - p^2) y = 0$$

dividing by x ,

$$\frac{d}{dx} \left(x \frac{d}{dx} \right) y - \frac{p^2}{x} y + 1 \cdot x \cdot y = 0$$

now we can identify

$$\begin{array}{l} \text{SL} \\ p(x) = x \\ \text{Sturm-Liouville} \end{array} \quad \begin{array}{l} s(x) = \\ p^2/x \\ \text{order of the function} \end{array} \quad \begin{array}{l} \lambda = 1 \\ w(x) = x \end{array}$$

Problem: obtain the normalization of the Bessel functions, using the change of variable where a are zeros of the Bessel function, i.e., $J_p(a) = 0$

Solution:

with the proposed transformation, we have

$$\frac{d}{dr} \left(r \frac{d}{dr} \right) y - \frac{p^2}{r} y + a^2 r y = 0$$

$p_{SL}(r) = r$ $s(r) = -\frac{p^2}{r}$ a $w(r) = r$

in this case, the interval is $0 \leq x \leq 1$
 $\rightarrow 0 \leq r \leq 1$
 because $J_p(a) = 0$.

From the PDF "Sturm Liouville" § Orthogonality... with weight function, we substitute $p_{SL}(r) = r$, $a_{SL} \rightarrow 0$, $b_{SL} \rightarrow 1$, $\psi_n(x) \rightarrow J_p(a r)$, $\psi_n(x) \rightarrow J_p(b r)$, $w(r) = r$

careful! chain rule!

$$r \left[J_p(a r) \frac{d}{dr} J_p(b r) - \frac{d}{dr} J_p(a r) J_p(b r) \right] \Big|_0^1 = (a^2 - b^2) \int_0^1 dr r J_p(a r) J_p(b r)$$

when we evaluate in $r=0$, the r cancels everything, so the surviving elements are

$$\frac{J_p(a) b J_p'(b) - a J_p'(a) J_p(b)}{a^2 - b^2} = \int_0^1 dr r J_p(a r) J_p(b r)$$

chain rule chain rule

if we take the limit $b \rightarrow a$, we get $0/0$.

We apply L'Hôpital's rule:

$$\begin{aligned} \int_0^1 dr r J_p(a r) J_p(b r) &= \lim_{b \rightarrow a} \frac{\frac{\partial}{\partial b} [J_p(a) b J_p'(b) - a J_p'(a) J_p(b)]}{\frac{\partial}{\partial b} (a^2 - b^2)} \\ &= \lim_{b \rightarrow a} \frac{J_p(a) \frac{d}{db} [b J_p(b)] - a J_p'(a) J_p'(b)}{-2b} \\ &= -a [J_p'(a)]^2 / (-2a) = \frac{1}{2} [J_p'(a)]^2 \end{aligned}$$

boundary condition

Now we use the recursion formula

$$\frac{d}{dx} \left(\frac{J_p(x)}{x^p} \right) = - \frac{J_{p+1}(x)}{x^p}$$

$$x = Kr \Rightarrow \frac{d}{dx} = \frac{1}{K} \frac{d}{dr}$$

$$\Rightarrow \frac{1}{K} \frac{d}{dr} \left(\frac{J_p(Kr)}{(Kr)^p} \right) = - \frac{J_{p+1}(Kr)}{(Kr)^p}$$

$$\frac{1}{K} \left[\frac{K J_p'(Kr)}{(Kr)^p} - \frac{J_p(Kr)}{(Kr)^{2p}} \cdot K \right] = - \frac{J_{p+1}(Kr)}{(Kr)^p}$$

at $r=1$, and applying bound. cond.,

$$\frac{1}{K} \frac{J_p'(K)}{K^p} = - \frac{J_{p+1}(K)}{K^p}$$

$$\Rightarrow J_p'(K) = -J_{p+1}(K)$$

so,

$$\int_0^1 dr \, r J_p(a r) J_p(b r) = \frac{1}{2} [J_{p+1}(a)]^2$$

Generalized Bessel ODE

Bessel functions form a family of solutions of the general ODE

$$y'' + \frac{1-2a}{x} y' + \left[(bcx^{c-1})^2 + \frac{a^2 - p^2 c^2}{x^2} \right] y = 0$$

with constant a, b, c, p

The solution of this equation is $y = x^a Z_p(b x^c)$

Z_p is J , N or a combination of them.

Problem: prove the recurrence relations

$$\frac{d}{dx}(x^{-n} Z_n) = -x^{-n} Z_{n+1} \quad ; \quad \frac{d}{dx}(x^n Z_n) = x^n Z_{n-1}$$

where Z_n is a solution of the Bessel equation.

This can be done in many ways. This is one:

Write the ODE in a factored form

$$\left(\frac{d}{dx} + \frac{n+1}{x} \right) \left(\frac{d}{dx} - \frac{n}{x} \right) Z_n = -Z_n$$

Let's define

$$\bar{Z}_n = \left(\frac{d}{dx} - \frac{n}{x} \right) Z_n$$

and if we apply the operator $\left(\frac{d}{dx} - \frac{n}{x} \right)$ on the factored ODE we form the expression

$$\left(\frac{d}{dx} - \frac{n}{x} \right) \underbrace{\left(\frac{d}{dx} + \frac{n+1}{x} \right) \left(\frac{d}{dx} - \frac{n}{x} \right) Z_n}_{\bar{Z}_n} = - \underbrace{\left(\frac{d}{dx} - \frac{n}{x} \right) Z_n}_{\bar{Z}_n}$$

$$\Rightarrow \left(\frac{d}{dx} - \frac{n}{x} \right) \left(\frac{d}{dx} + \frac{n+1}{x} \right) \bar{Z}_n = -\bar{Z}_n$$

$$\Rightarrow \frac{d^2}{dx^2} \bar{Z}_n + \frac{1}{x} \frac{d}{dx} \bar{Z}_n + \left[1 - \frac{(n+1)^2}{x^2} \right] \bar{Z}_n = 0$$

This is the Bessel ODE of order $n+1$

$$\Rightarrow \left(\frac{d}{dx} - \frac{n}{x} \right) Z_n = \bar{Z}_n = -Z_{n+1}$$

$$\Rightarrow \left[\frac{dZ_n}{dx} - \frac{n}{x} Z_n = -Z_{n+1} \right] \cdot x^{-n}$$

$$\Rightarrow \frac{d}{dx} (x^{-n} Z_n) = -x^{-n} Z_{n+1}$$

For the second identity, we try another way, using the explicit series of $J_p(x)$

$$\begin{aligned} \frac{d}{dx} [x^p J_p(x)] &= \frac{d}{dx} \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p+1)} \frac{x^{2n+2p}}{2^{2n+p}} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+2p)}{\Gamma(n+1)\Gamma(n+p+1)} \frac{x^{2n+2p-1}}{2^{2n+p}} \end{aligned}$$

but $\Gamma(n+p+1) = (n+p)\Gamma(n+p)$, so we use this to cancel the $(n+p)$ in the numerator:

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p)} \frac{x^{2n+2p-1}}{2^{2n+p-1}} \quad \rightarrow \text{the 2 in the numerator}$$

dividing both sides by x^p

$$\frac{1}{x^p} \frac{d}{dx} [x^p J_p(x)] = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p)} \left(\frac{x}{2} \right)^{2n+p-1} = J_{p-1}(x)$$

Problem: verify that the Bessel ODE

$$x^2 y'' + x y' + (x^2 - n^2) y = 0$$

can be factored as

$$\left(\frac{d}{dx} + \frac{n+1}{x} \right) \left(\frac{d}{dx} - \frac{n}{x} \right) Z_n = -Z_n$$

Solution:

We apply the first operator,

$$\left(\frac{d}{dx} + \frac{n+1}{x} \right) \left[\frac{dZ_n}{dx} - \frac{nZ_n}{x} \right] = -Z_n$$

We apply the second operator,

$$\frac{d^2 Z_n}{dx^2} - \frac{d}{dx} \left[\frac{nZ_n}{x} \right] + \frac{(n+1)}{x} \frac{dZ_n}{dx} - \frac{(n+1)n}{x^2} Z_n = -Z_n$$

$$\frac{d^2 Z_n}{dx^2} - \frac{n}{x} \frac{dZ_n}{dx} + \frac{nZ_n}{x^2} + \frac{(n+1)}{x} \frac{dZ_n}{dx} - \frac{(n+1)n}{x^2} Z_n = -Z_n$$

$$\frac{d^2 Z_n}{dx^2} + \frac{1}{x} \frac{dZ_n}{dx} [-n + n+1] + \frac{Z_n}{x^2} [1 - n - 1] n = -Z_n$$

$$\left[Z_n'' + \frac{1}{x} Z_n' - \frac{Z_n}{x^2} n^2 = -Z_n \right] x^2$$

$$x^2 Z_n'' + x Z_n' - Z_n n^2 + Z_n x^2 = 0 \quad \checkmark$$

Copy of the example in the lecture notes

Example: Schlöfli integral for non-polynomials and its relation to generating functions.

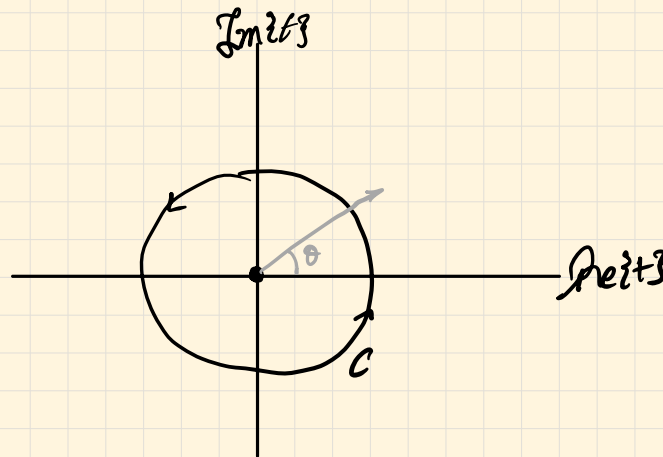
The Bessel functions of the first kind are not polynomials, but they have a generating function $g(x, t) = e^{(x/2)(t-1/t)} = \sum_n J_n(x) t^n$ (see table). If we apply the residue theorem to an integral containing the generating function, we get

$\oint_C \frac{e^{(x/2)(t-1/t)}}{t^{n+1}} dt = \oint_C \sum_m J_m(x) t^{m-n-1} = 2\pi i J_n(x)$ with C encircling $t = 0$. With the change of variable $t = e^{i\theta} \implies e^{(x/2)(t-1/t)} = e^{ix \sin \theta}$, and C being the unit circle,

we find $2\pi i J_n(x) = \int_0^{2\pi} e^{i(x \sin \theta - n\theta)} i d\theta$ in general for $x \in \mathbb{C}$ or

$J_n(x) = \frac{1}{\pi} \int_0^\pi \cos(x \sin \theta - n\theta) d\theta$ for $x \in \mathbb{R}$. In this case, the generating function

was used to get (a) Schlöfli representation, but we can also derive generating functions (for example, for polynomials) using the/a Schlöfli representation.



Problem. In a diffraction problem for a circular aperture, the integral

$$\Phi = \int_0^a \int_0^{2\pi} e^{ibr \cos \theta} d\theta r dr$$

appears, where a, b are constants. Evaluate the integral using Bessel's functions.

The integral representation of the Bessel functions is

$$J_n(x) = \frac{1}{\pi} \int_0^\pi \cos(x \sin \theta - n\theta) d\theta \quad \text{or} \quad J_n(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{i(x \sin \theta - n\theta)} d\theta$$

for $n=0$,

$$J_0(x) = \frac{1}{\pi} \int_0^\pi \cos(x \sin \theta) d\theta = \frac{1}{2\pi} \int_0^{2\pi} \cos(x \sin \theta) d\theta =$$

$\cos(x \sin \theta)$ repeats itself
in all four quadrants

$$= \frac{1}{2\pi} \int_0^{2\pi} e^{ix \sin \theta} d\theta = \frac{1}{2\pi} \int_0^{2\pi} e^{ix \cos \theta} d\theta$$

because of Euler's formula and $\int_0^{2\pi} \sin(x \sin \theta) d\theta = 0$

then

$$\Phi = 2\pi \int_0^a J_0(br) r dr$$

using the recurrence formula $\frac{d}{dx}(x^n J_n(x)) = x^n J_{n-1}(x)$ with $n=1$,

$$\Phi = \frac{2\pi a b}{b^2} J_1(ab).$$

Problem. Evaluate the integral

$$\int_0^\infty e^{-ax} J_0(bx) dx, \quad a > 0, b > 0.$$

Solution. Introduce the integral representation of the Bessel function

$$= \int_0^\infty e^{-ax} dx \frac{2}{\pi} \int_0^{\pi/2} \cos(bx \sin \varphi) d\varphi$$

exchanging the integration order

$$= \frac{2}{\pi} \int_0^{\pi/2} d\varphi \int_0^\infty e^{-ax} \cos(bx \sin \varphi) dx$$

integrating,

$$= \frac{2}{\pi} \int_0^{\pi/2} d\varphi \frac{a}{a^2 + b^2 \sin^2 \varphi} = \frac{1}{\sqrt{a^2 + b^2}} \quad \square$$

Problem. John Bernoulli needed to solve the following non-linear differential equation:

$$y' = x^2 + y^2$$

but he couldn't. His brother and rival James Bernoulli could solve it by transforming it into a linear second order ODE with the change of variable

$$y = -\frac{1}{u} \frac{du}{dx}$$

into

$$\Rightarrow u'' + x^2 u = 0$$

Solve the equation for $u(x)$ by further transforming it into a Bessel equation.

Solution:

Generalized Bessel ODE

$$u'' + \frac{1-2a}{x} u' + \left[(bcx^{c-1})^2 + \frac{a^2 - p^2 c^2}{x^2} \right] u = 0$$

we want:

$$\begin{aligned} \underbrace{1-2a}_{=0} &= 0 \\ \Rightarrow 1-2a &= 0 \\ \Rightarrow a &= 1/2 \end{aligned}$$

$$\begin{aligned} \underbrace{a^2 - p^2 c^2}_{=0} &= 0 \\ \Rightarrow \left(\frac{1}{2}\right)^2 - p^2 c^2 &= 0 \\ \Rightarrow p^2 c^2 &= \frac{1}{4} \end{aligned}$$

$$\begin{aligned} &= x^2 \\ \text{we need } c &= 2 \\ \Rightarrow (bx)^2 & \\ \text{we need } b &= 1/2 \\ \Rightarrow x^2 & \checkmark \end{aligned}$$

$$p^2 \cdot 4 = \frac{1}{4}$$

$$\Rightarrow p^2 = \frac{1}{16} \Rightarrow p = \pm \frac{1}{4}$$

$$\Rightarrow u(x) = x^a Z_p(bx^c) = \sqrt{x} Z_4\left(\frac{x^2}{2}\right)$$

$$u(x) = c_1 \sqrt{x} J_4(x^2/2) + c_2 \sqrt{x} Y_4(x^2/2) \quad \blacksquare$$

Problem. The solutions of the second order differential equation

$$u'' - xu = 0$$

are called the Airy functions. Solve the differential equation by first transforming it into a generalized Bessel ODE.

Solution:

Generalized Bessel ODE

$$u'' + \frac{1-2a}{x} u' + \left[(bcx^{c-1})^2 + \frac{a^2 - p^2 c^2}{x^2} \right] u = 0$$

we want = 0
 $\Rightarrow a = \frac{1}{2}$

we want = 0
 $\Rightarrow p^2 c^2 = \frac{1}{4}$

we want = -x
 $\Rightarrow c = \frac{3}{2}$

$p^2 \cdot \frac{9}{4} = \frac{1}{4}$

$\Rightarrow \left(b \cdot \frac{3}{2} \cdot x^{\frac{1}{2}} \right)^2 = -x \Rightarrow p = \pm \frac{1}{3}$

$\Rightarrow b^2 \cdot \left(\frac{3}{2} \right)^2 x = -x$

$\Rightarrow b^2 \cdot \left(\frac{3}{2} \right)^2 = -1 \Rightarrow b = \frac{2}{3} i$

$$\Rightarrow u(x) = x^a Z_p(bx^c) = x^{\frac{1}{2}} Z_{\frac{1}{3}}\left(\frac{2}{3} i x^{\frac{3}{2}}\right)$$

in terms of the Modified Bessel functions (Bessel of imaginary argument)

$$u(x) := c_1 Ai(x) + c_2 Bi(x)$$

$$Ai(x) = \frac{x^{\frac{1}{2}}}{3} \left[I_{-\frac{1}{3}}\left(\frac{2x^{\frac{3}{2}}}{3}\right) - I_{\frac{1}{3}}\left(\frac{2x^{\frac{3}{2}}}{3}\right) \right] = \frac{1}{\pi} \left(\frac{x}{3}\right)^{\frac{1}{6}} K_{\frac{1}{3}}\left(\frac{2x^{\frac{3}{2}}}{3}\right)$$

$$Bi(x) = \left(\frac{x}{3}\right)^{\frac{1}{6}} \left[I_{-\frac{1}{3}}\left(\frac{2x^{\frac{3}{2}}}{3}\right) + I_{\frac{1}{3}}\left(\frac{2x^{\frac{3}{2}}}{3}\right) \right]$$

Problem: show that $j_0(x) = \frac{\sin x}{x}$

starting from the series expansion of $J_{1/2}(x)$.

Solution:

$$J_{1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1) \Gamma(n+3/2)} \left(\frac{x}{2}\right)^{2n+1/2}$$

$$\begin{aligned} \Gamma(1/2) &= \sqrt{\pi} \\ \Gamma(n+1) &= n! \\ \Gamma(n+1) n \Gamma(n) \end{aligned}$$

we know that

$$\begin{aligned} \sin x &= \underbrace{x}_{n=0} - \underbrace{\frac{x^3}{3!}}_{n=1} + \underbrace{\frac{x^5}{5!}}_{n=2} - \underbrace{\frac{x^7}{7!}}_{n=3} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{\underbrace{(2n+1)!}_m} x^{2n+1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(2n+2)} x^{2n+1} \end{aligned}$$

$$2^{2n+1/2} = 2^{\frac{(2n+2)-1-1/2}{2(n+1)}}$$

$$\Rightarrow J_{1/2}(x) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+1) \Gamma(n+1+1/2)} \frac{(-1)^n x^{2n+1-1/2}}{2^{2(n+1)-1} \cdot 2^{-1/2}}$$

$$2^{2x-1} \Gamma(x) \Gamma(x+1/2) = \sqrt{\pi} \Gamma(2x)$$

Tables
Gamma function
duplication formula

$$= \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi} \Gamma(2n+2)} \underbrace{(-1)^n x^{2n+1}}_{\sin(x)} \cdot \underbrace{x^{-1/2}}_{\frac{1}{2^{-1/2}}}$$

doesn't depend
on x

$$= \sqrt{\frac{2}{\pi}} x^{-1/2} \sin x$$

$$= \sqrt{\frac{2}{\pi x}} \sin x$$

finally,

$$\begin{aligned} \Rightarrow j_0(x) &= \sqrt{\frac{\pi}{2x}} J_{1/2}(x) \\ &= \sqrt{\frac{\pi}{2x}} \sqrt{\frac{2}{\pi x}} \sin x \\ &= \frac{\sin x}{x} \end{aligned}$$

Problem: use the recurrence formula

$$\frac{d}{dx} [x^{-p} J_p(x)] = -x^{-p} J_{p+1}(x)$$

to verify the Rayleigh's formula for spherical Bessel functions:

$$j_n(x) = x^n \left(-\frac{1}{x} \frac{d}{dx} \right)^n \left(\frac{\sin x}{x} \right)$$

For the case $n = 0$, $n=1$.

Solution:

From the recurrence formula, we know that

$$x^{-1/2} J_{3/2}(x) = -\frac{d}{dx} [x^{-1/2} J_{1/2}(x)]$$

We also know that

$$j_0(x) = \sqrt{\frac{\pi}{2x}} J_{1/2}(x) = \frac{\sin x}{x}$$

$$\begin{aligned} \Rightarrow x^{-1/2} J_{3/2}(x) &= -\frac{d}{dx} \left[\sqrt{\frac{2}{\pi}} j_0(x) \right] \\ &= -\sqrt{\frac{2}{\pi}} \frac{d}{dx} \left(\frac{\sin x}{x} \right) \end{aligned}$$

definition:

$$j_n(x) = \sqrt{\frac{\pi}{2x}} J_{(2n+1)/2}(x)$$

From the definition,

$$j_1(x) = \sqrt{\frac{\pi}{2}} x^{-1/2} J_{3/2}(x)$$

$$\Rightarrow \sqrt{\frac{2}{\pi}} j_1(x) = -\sqrt{\frac{2}{\pi}} \frac{d}{dx} \left(\frac{\sin x}{x} \right)$$

by induction using the recurrence formula, we can also do it for the general case.

Problem. Show that the Ansatz $\psi(z) = e^{-\frac{z^2}{2}} v(z)$ on the differential equation

$$\frac{d^2}{dz^2} \psi(z) + \left(\frac{2E}{\hbar\omega} - z^2 \right) \psi(z) = 0 \quad (1)$$

with $E, \hbar, \omega$ being constants, transforms it into the Hermite ODE.

Solution:

$$\psi(z) = e^{-z^2/2} v(z) \quad (2)$$

$$\frac{d\psi}{dz} = -z e^{-z^2/2} v(z) + e^{-z^2/2} \frac{dv(z)}{dz} = e^{-z^2/2} \left(-zv + \frac{dv}{dz} \right)$$

$$\begin{aligned} \frac{d^2\psi}{dz^2} &= -z e^{-z^2/2} \left(-zv + \frac{dv}{dz} \right) + e^{-z^2/2} \left(-v - z \frac{dv}{dz} + \frac{d^2v}{dz^2} \right) \\ &= e^{-z^2/2} \left(z^2 v - 2z \frac{dv}{dz} - v + \frac{d^2v}{dz^2} \right) \quad (3) \end{aligned}$$

Substituting (3), (2) in (1), we see that the factor $e^{-z^2/2}$ cancels out:

$$\begin{aligned} z^2 v - 2z \frac{dv}{dz} - v + \frac{d^2v}{dz^2} + \frac{2E}{\hbar\omega} v - z^2 v &= 0 \\ \Rightarrow v'' - 2zv' + \left(\frac{2E}{\hbar\omega} - 1 \right) v &= 0 \quad \checkmark \end{aligned}$$

Problem: Show the recurrence formula

$$H'_n(x) = 2n H_{n-1}(x)$$

Solution:

We start from the generating function

$$\Phi(x, h) = \sum_{n=0}^{\infty} H_n(x) \frac{h^n}{n!} = e^{-h^2 + 2hx}$$

And try $\frac{\partial \Phi}{\partial x} = 2h e^{-h^2 + 2hx} = 2h \Phi = 2h$

We substitute Φ and $\frac{\partial \Phi}{\partial x}$ by the series.

$$\Phi(x, h) = \sum_{n=0}^{\infty} H_n(x) \frac{h^n}{n!}; \quad \frac{\partial \Phi}{\partial x} = \sum_{n=0}^{\infty} H'_n(x) \frac{h^n}{n!}$$

$$\sum_{n=0}^{\infty} H'_n(x) \frac{h^n}{n!} = \sum_{n=0}^{\infty} 2h H_n(x) \frac{h^n}{n!}$$

$$\downarrow$$

$$2 \sum_{n=0}^{\infty} H_n(x) \frac{h^{n+1}}{n!}$$

$$\downarrow \text{ let } m = n+1 \Rightarrow \text{when } n=0, m=1$$

$$\sum_{n=0}^{\infty} H'_n(x) \frac{h^n}{n!} = 2 \sum_{m=1}^{\infty} H_{m-1}(x) \frac{h^m}{(m-1)!}$$

$\downarrow m=n$, recursivity of factorial

$$\sum_{m=0}^{\infty} H'_m(x) \frac{h^m}{m(m-1)!} = 2 \sum_{m=1}^{\infty} H_{m-1}(x) \frac{h^m}{(m-1)!}$$

Finally, for $m \geq 1$, $\frac{H'_m(x)}{m} = 2 H_{m-1}(x) \Rightarrow H'_n(x) = 2n H_{n-1}(x) \quad \checkmark$

Problem: Write an integral representation of the Hermite polynomials.

Solution:

Start from the generating function

$$g(x,t) = e^{-t^2+2tx} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}$$

and multiply by t^{-m-1} :

$$t^{-m-1} e^{-t^2+2tx} = \sum_{n=0}^{\infty} t^{-m-1+n} \frac{H_n(x)}{n!}$$

integrate in the complex plane around the origin:

$$\oint t^{-m-1} e^{-t^2+2tx} dt = \oint \sum_{n=0}^{\infty} t^{-m-1+n} \frac{H_n(x)}{n!} dt$$

With poles at $t=0$, we find with the residue theorem that only the term $n=m$ survives:

$$\oint t^{-m-1} e^{-t^2+2tx} dt = 2\pi i \frac{H_m(x)}{m!}$$

$$\Rightarrow H_m(x) = \frac{m!}{2\pi i} \oint t^{-m-1} e^{-t^2+2tx} dt$$

Problem: find the normalization of the Hermite polynomials.

Solution:

We start this time from the generating function (we multiply two of them, one with the variable t and one with the variable s) and build the following expression

$$e^{-x^2} e^{-s^2+2sx} e^{-t^2+2tx} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} e^{-x^2} H_m(x) H_n(x) \frac{s^m t^n}{m! n!}$$

now we integrate over the interval $]-\infty, \infty[$ and from the orthogonality, we know the terms other than $m=n$ will be zero.

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(st)^n}{n! n!} \int_{-\infty}^{\infty} e^{-x^2} [H_n(x)]^2 dx &= \int_{-\infty}^{\infty} e^{-x^2-s^2+2sx} e^{-t^2+2tx} e^{-2st+2st} dx \\ &= \int_{-\infty}^{\infty} e^{-(x-s-t)^2} e^{2st} dx = \sqrt{\pi} e^{2st} = \sqrt{\pi} \sum_{n=0}^{\infty} \frac{2^n (st)^n}{n!} \end{aligned}$$

Known integral

Examining coefficients term by term of the power st we obtain

$$\int_{-\infty}^{\infty} e^{-x^2} [H_n(x)]^2 dx = 2^n \sqrt{\pi} n!$$

Problem: show that the differential operator of the Hermite ODE

can be factorized. $y'' - 2xy' + 2ny = 0$

Use this factorization to verify that $H'_n(x) \propto H_{n-1}(x)$, in agreement with one of the recursion relations.

Solution:

We write $\left(\frac{d^2}{dx^2} - 2x\frac{d}{dx}\right)y = -2ny$

let $\hat{D} = \frac{d}{dx}$.

$$\Rightarrow (\hat{D}^2 - 2x\hat{D})y = -2ny$$

Now it's simple to factor this as $(\hat{D} - 2x)\hat{D}y = -2ny$.

Applying the first factor means that

$$(\hat{D} - 2x)H'_n = -2nH_n$$

$$(\hat{D} - 2x)DH_{n-1} = -2(n-1)H_{n-1}$$

That is, the differential operator on the left brings out the order of the polynomial on the right.

We differentiate the Hermite equation on both sides

$$D(D - 2x)H'_n = -2nH'_n$$

$$\Rightarrow D^2 - D(2xH'_n) = -2nH'_n$$

$$\Rightarrow D^2 - 2H'_n - 2xDH'_n = -2nH'_n$$

$$\Rightarrow (D - 2x)DH'_n = -2(n-1)H'_n$$

From our earlier conclusion, we see that H'_n brings out an order $n-1$, so it means that

$$H'_n \propto H_{n-1}$$

in order to bring that order of the polynomial out. We would need another equation to determine the constant of proportionality.

Problem: expand the function

$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 < x < 1 \\ 0, & x > 1 \end{cases} = H(x) - H(x-1)$$

in terms of a Laguerre series.

Solution: we need to compute the coefficients

$$c_n = \langle f(x), L_n(x) \rangle$$

We use Rodrigues's formula

$$c_n = \int_0^1 e^{-x} \frac{1}{n!} \frac{d^n}{dx^n} (x^n e^{-x}) dx$$

$$= \frac{1}{n!} \left. \frac{d^{n-1}}{dx^{n-1}} (x^n e^{-x}) \right|_0^1 = \frac{1}{n!} \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{d^{n-1-k}}{dx^{n-1-k}} (x^n) \frac{d^k}{dx^k} (e^{-x}) \Big|_0^1$$

$$= \frac{1}{n!} \sum_{k=0}^{n-1} \frac{(n-1)!}{k! (n-1-k)!} \underbrace{n(n-1) \dots (n-1-k)}_{n-1-k \text{ veces}} x^{k+1} (-1)^k e^{-x} \Big|_0^1$$

$$= \sum_{k=0}^{n-1} \frac{(n-1)!}{n(n-1)! (n-1-k)! k!} \frac{n!}{(k+1)!} x^{k+1} (-1)^k e^{-x} \Big|_0^1$$

$$= \sum_{k=0}^{n-1} \frac{n!}{n(k+1)(n-1-k)! [k!]^2} (-1)^k \frac{1}{e}$$

$$\Rightarrow f(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{n-1} \frac{n! (-1)^k}{n(k+1)(n-1-k)! [k!]^2} \frac{1}{e} L_n(x)$$

Leibniz diff. rule:

$$\frac{d^n}{dx^n} (fg) = \sum_{k=0}^n \binom{n}{k} \frac{d^{n-k}}{dx^{n-k}} f \frac{d^k}{dx^k} g$$

$$\binom{n}{k} = \frac{n(n-1)(n-2) \dots (n-k+1)}{k!}$$

$$= \frac{n!}{k! (n-k)!}$$

$$\frac{d^p}{dx^p} (x^n) = n(n-1) \dots \underbrace{(n-p+1)}_{p \text{ veces}} x^{n-p} = \frac{n!}{(n-p)!} x^{n-p}$$