Notes on

Mathematical Physics

Fourier series

Mathematical review

Field (Körper, cuerpo): a set $\mathbb{K}$ is a field when an addition $\mathbb{K} \times \mathbb{K} \to \mathbb{K}, (x, y) \to x + y$ and multiplication $\mathbb{K} \times \mathbb{K} \to \mathbb{K}, (x, y) \to xy$ can be defined in such a way that both operations are associative, commutative and there are neuter and inverse elements. (Examples: $\mathbb{R}, \mathbb{Q}, \mathbb{C}$). Don't confuse this definition of a "field" with a scalar or vector field. In this document, $\mathbb{N}_0$ is the set $\mathbb{N} \cup \{0\}$.

Function: a function *f* from a set A to a set B is a correspondence such that for each element $x \in A$ there is exactly one corresponding element $y \in B, y = f(x)$. For example, A and B can be $\mathbb{R}$, $\mathbb{R}^n$or $\mathbb{C}$. Example of a function definition: $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$.

Comment: We have to be careful in defining the domain of a function; also with programming (we must tell the compiler whether the arguments are real or complex, and the function return is real or complex). Example of the function $f : \mathbb{R} \to \mathbb{C}, f(x) = i + x$ in Fortran:

```
function f(x)
      real x
      complex f
      f = (0,1) + x
end function
```

Vector space: a set V is a vector space over a field $\mathbb{K}$ if there is an addition $V \times V \to V : (x, y) \to x + y$ and a scalar multiplication $\mathbb{K} \times V : (\lambda, x) \to \lambda x$. The scalar multiplication should satisfy: a) $\lambda(x + y) = \lambda x + \lambda y$; b) $(\lambda + \mu)x = \lambda x + \mu x$; c) $(\lambda\mu)x = \lambda(\mu x)$; d) $1x = x$, for all $\lambda, \mu \in \mathbb{K}$ and $x, y \in V$.

Comment: the definition of a vector space does not define what a vector is. Several mathematical objects over a given field can be behave as a vector (not only an "arrow" linking two points in physical space is a vector). Polynomials and functions can also form vector spaces (see below), as long as they satisfy the properties of a vector space.

Polynomial: a real polynomial is a function $f : \mathbb{R} \to \mathbb{R}$ of the form $f(x) = a_0 + a_1x + \cdots + a_nx^n = \sum_{k=0}^{n} a_kx^k$. $a_k \in \mathbb{R}$ are the coefficients of the polynomial.

Linear independence: The vectors $\psi_1, \psi_2, \ldots$ are linearly independent when the equation $\sum c_n\psi_n = 0$ only has the trivial solution $c_1 = c_2 = \cdots = 0$. The maximum number of linearly independent vectors is called the *dimension* of the vector space.

Basis and coordinates: there are linearly independent vectors $\{e_1, e_2, \ldots\} \subset V$ for which each vector $v \in V$ can be written as a linear combination of basis vectors: $v = \sum_i x_ie_i$. There is always a bijective transformation between two given bases. The x_i are the coordinates of the vector, usually written as a column vector (or transposed row vector to save space). Example: polynomials $f(X)$ can form a vector space; for the polynomial $f(X) = 1 + 2X + 3X^2$, the coordinates in the basis $\{1, X, X^2\}$ are $(1,2,3)^T$, but in the basis $\{1, 1 + X, 1 + X + X^2\}$, they are $(-1, -1, 3)^T$.

Comment: continuing with the previous example, we see that for the set (sequence) of linearly-independent polynomials $\{1, X, X^2, \ldots, X^n, \ldots\}$ (until infinity), we obtain a vector space (of functions) of infinite dimensions.

Sequence: it is a function $a: \mathbb{N}_0 \to \mathbb{R}$ (real sequence) or $a: \mathbb{N}_0 \to \mathbb{C}$ (complex sequence), such that one obtains the set $\{a_0, a_1, \ldots, a_n, \ldots\}$ for $n \in \mathbb{N}_0$. Some sequences converge to a limit value $a_\infty \in \mathbb{R}$ for $\lim_{n\to\infty} a_n = a_\infty$.

Series: one can do a partial sum of the elements of a sequence (in numpy, for an array: `np.cumsum()`):

$$\begin{array}{llllll}\{a_0, & a_1, & a_2, & \ldots, & a_n & ,\ldots, a_\infty \quad \} \\ \{a_0, & a_0 + a_1, & a_0 + a_1 + a_2, & \ldots, & \sum_{k=0}^{n} a_k & ,\ldots, \sum_{k=0}^{\infty} a_k\}\end{array}$$

the last element of the partial sum (the sum of the all the elements of the sequence up to infinity) is the series. Some series converge to a given value (i.e., the last element of the partial sum $\in \mathbb{R}$ for a real series).

Periodic function: a function $f: \mathbb{R} \to \mathbb{C}$ is periodic with period $T > 0$ if $f(x + T) = f(x)$ for all $x \in \mathbb{R}$. Examples: $\sin(x)$, e^{ix} are periodic functions with period 2π. The function $\sin(2\pi k x / T)$ is T-periodic for $T > 0, k \in \mathbb{Z}$.

How to change the period: if $f(x)$ has period T_1, $g(x) = f(T_1 x / T_2)$ has period T_2. Exercise: show this! (hint: use the definition of periodic function).

Inner product of two periodic functions: $\langle f, g \rangle = K \int_{-T/2}^{T/2} f(x)\bar{g}(x)dx$, given functions $f, g: \mathbb{R} \to \mathbb{C}$ with period $T > 0$ and a normalization factor K. The bar means complex conjugation. Similar to other vector fields (like forces or velocities), two functions can be orthogonal when their scalar product is zero, and orthonormal if $\langle f, g \rangle = \begin{cases} 1 & \text{if } f = g \\ 0 & \text{otherwise} \end{cases}$. One can define the normalization factor K if needed using the orthonormalization condition.

Series expansion of a function: one can develop a function $f(x)$ into a series (i.e., set the function as the limit value to which the series must converge) of a set of orthonormal periodic functions $\{g_k(x)\}$ ($k \in \mathbb{Z}$) that form the orthonormal basis of a vector space:

$$f(x) = \sum_k c_k g_k(x).$$

The limit of the series can be, e.g., from $k = -\infty$ to ∞, or from $k = 0$ to ∞. The coefficients $c_k \in \mathbb{C}$ can be found using the inner product (multiplying from the right by $g_\ell(x)$ with a different index $\ell \in \mathbb{Z}$):

$$\langle f(x), g_\ell(x) \rangle = \sum_k c_k \langle g_k(x), g_\ell(x) \rangle = \sum_k c_k \delta_{\ell k} \quad (\delta_{\ell k} \text{ is the Kronecker delta}).$$

$$\implies c_\ell = \langle f(x), g_\ell(x) \rangle.$$

Real Fourier series in cosines and sines: one can show (exercise!) that the set $\{1/\sqrt{2}, \cos(x), \cos(2x), \ldots, \sin(x), \sin(2x), \ldots\}$ forms an orthonormal basis of a vector space of periodic functions with period 2π within the interval $[-\pi, \pi[$ and with a normalization factor $K = 1/\pi$. Then, one can expand a function $f: \mathbb{R} \to \mathbb{R}$ as

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx) \quad \text{with } k \in \mathbb{N}$$

The coefficients $a_k, b_k \in \mathbb{R}$ can be then computed as

$$a_0 = \langle f, 1/\sqrt{2} \rangle, \qquad a_k = \langle f, \cos(kx) \rangle\,, \qquad b_k = \langle f, \sin(kx) \rangle.$$

Real Fourier series for an arbitrary period: In general, for a periodic function $f : [-T/2, T/2[\to \mathbb{R}$ with period T, the Fourier series defined using the basis functions $\{\sin(2\pi k x/T), \cos(2\pi k x/T)\}$ with $k \in \mathbb{N}$ is

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(2\pi k x/T) + b_k \sin(2\pi k x/T)$$

The coefficients $a_k, b_k \in \mathbb{R}$ can be then computed as

$$a_k = \langle f, \cos(2\pi k x/T)\rangle \ \ (k \in \mathbb{N}_0), \qquad b_k = \langle f, \sin(2\pi k x/T)\rangle \ \ (k \in \mathbb{N})$$

with the normalization constant $K = 2/T$.

- Simple theory exercise: shift the function definition from $[-T/2, T/2[$ to $[0, T[$.

Complex Fourier series: The set of functions $\{e^{2\pi i k x/T}\}$ (each $[-T/2, T/2[\to \mathbb{C}$, $k \in \mathbb{Z}$) forms an orthonormal basis of a vector space with a period T and a normalization factor $K = 1/T$. We define the Fourier series expansion of a function $f : [-T/2, T/2[\to \mathbb{C}$ of period T as

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k x/T}$$

and then, the coefficients $c_k \in \mathbb{C}$ are computed as $c_k = \langle f(x), e^{2\pi i k x/T}\rangle = \frac{1}{T}\int_{-T/2}^{T/2} f(x) e^{-2\pi i k x/T} dx$ (remember to do the complex conjugate of the expontential!).

- Comment: for a real function $f : [-T/2, T/2[\to \mathbb{R}$, the coefficients c_k can still be complex, but their product with the basis functions should yield a real number in the end.
- Simple exercise: What is the relation between the coefficients of the Fourier series of $f(x)$ and the Fourier series of the shifted function $f(x+a)$ $(a \in \mathbb{R})$?

Relations between a real and complex Fourier series: for a function $f : \mathbb{R} \to \mathbb{C}$ of period T, we can expand in both bases:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(2\pi k x/T) + b_k \sin(2\pi k x/T) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k x/T}$$

but this time, $a_k, b_k, c_k \in \mathbb{C}$ in general. If the $c_k, k \in \mathbb{Z}$ are given:

$$a_0 = 2c_0, \quad a_k = c_k + c_{-k}, \quad b_k = i(c_k - c_{-k}) \quad \text{for } k \in \mathbb{N}.$$

If $a_0, a_k, b_k, k \in \mathbb{N}$ are given:

$$c_0 = a_0/2, \quad c_k = \frac{1}{2}(a_k - i b_k), \quad c_{-k} = \frac{1}{2}(a_k + i b_k) \text{ for } k \in \mathbb{N}.$$

Handling of discontinuities: consider $f : [-T/2, T/2[\to \mathbb{R}$ but with points $\alpha_1, \ldots, \alpha_\ell$ within the domain where f is discontinuous. Then, the Fourier expansion $f_F(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx)$ can be related as $f(x) = f_F(x)$ if f is continuous in x, but if it is not, then for each $x = \alpha_k$ $(k = 1, \ldots, \ell)$, $f_F(\alpha_k) = \frac{f(\alpha_k^-) + f(\alpha_k^+)}{2}$ (where α_k^+ and α_k^- are the values from the left and right sides of the discontinuity)

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