Funciones Gamma, Beta y otras especiales

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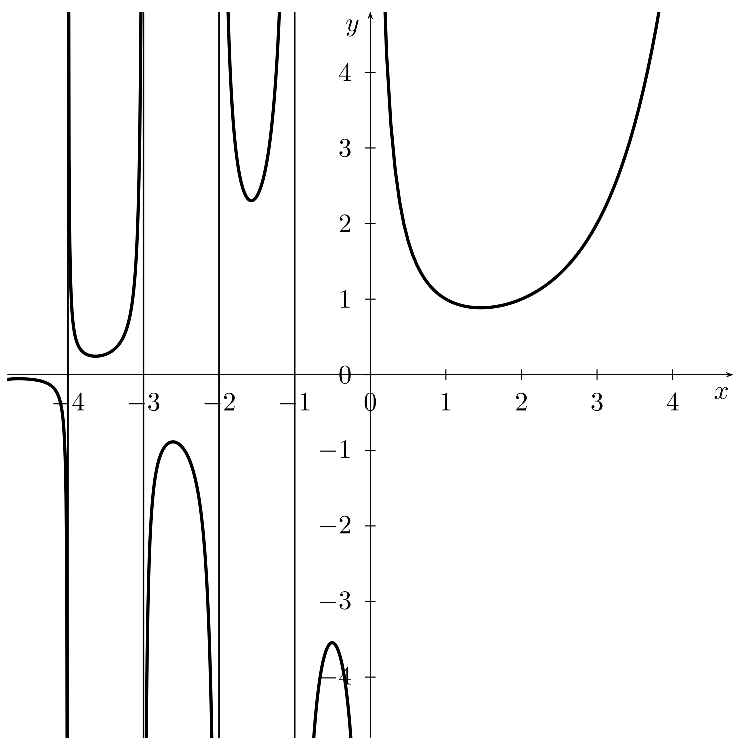
September 26, 2025

Función Gamma (Euler)

Para $n > 0$, $\Re(n) > 0$:

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx$$



Fórmula de recurrencia

$$\Gamma(n+1) = n\Gamma(n)$$
$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Fórmula Asintótica (n grande)

$$\Gamma(n+1) = \sqrt{2\pi n}\, n^n e^{-n} e^{\Theta/(12(n+1))}, \quad 0 < \Theta < 1$$
$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \quad \text{(Aproximación factorial de Stirling)}$$

Propiedades

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!\, n^x}{x(x+1)\cdots(x+n)}$$
$$\Gamma(x+1) = x\Gamma(x)$$
$$\Gamma(n+1) = n! \quad \text{(n entero)}$$
$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$$
$$\Gamma(x+n) = \prod_{k=0}^{n-1}(x+k)\Gamma(x)$$

Fórmula de la duplicación de Legendre

$$2^{2x-1}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right) = \sqrt{\pi}\,\Gamma(2x)$$

Derivada de la función Gamma

$$\frac{d}{dz}\Gamma(z+1) = \Gamma(z) + z\frac{d\Gamma}{dz} = \int_0^\infty x^{z-1}e^{-x}\ln t\, dx$$
$$\frac{d^n}{dz^n}\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}[\ln t]^n\, dx$$

Cálculo de $\Gamma\left(\frac{1}{2}\right)$

Sea $t = u^2$, $dt = 2u\,du$:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt = 2\int_0^\infty u^{2z-1} e^{-u^2} du$$

Para $z = \frac{1}{2}$:

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-u^2} du = \int_{-\infty}^\infty e^{-u^2} du$$

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^\infty \int_0^\infty e^{-(u^2+v^2)} du\,dv$$

Cambio a coordenadas polares: $u = \rho\cos\varphi$, $v = \rho\sin\varphi$, $du\,dv = \rho\,d\rho\,d\varphi$

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^\infty \int_0^{\pi/2} e^{-\rho^2} \rho\,d\rho\,d\varphi = 2\pi\int_0^\infty \rho e^{-\rho^2} d\rho = -\left. e^{-\rho^2}\right|_0^\infty = \pi$$

$$\Rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Problema 5.9.7 Arfken–Weber (Quinta edición)

Este problema trata de la radiación de cuerpo negro.
Ley de Planck

$$u_\nu(T) = 8\pi\frac{\nu^3}{c^3}\frac{1}{e^{h\nu/K_BT} - 1}$$

La energía total irradiada de un cuerpo negro es (Ley de Stefan-Boltzmann)

$$u = \frac{8\pi K_B T^4}{c^3h^3}\int_0^\infty \frac{x^3}{e^x - 1} dx = \sigma T^4, \qquad \sigma = \frac{2\pi^5 K_B^4}{15c^2h^3}$$

$$I = \int_0^\infty \frac{x^3}{e^x - 1} dx = 3!\zeta(4), \qquad \zeta(4) = \frac{\pi^4}{90} \quad (\zeta(s) \text{ Zeta de Riemann}).$$

Veamos de donde sale este resultado

$$I_n = \int_0^\infty \frac{x^n}{e^{\alpha x} - 1}\,dx$$

Sea $y = \alpha x$:

$$I_n = \frac{1}{\alpha^{n+1}}\int_0^\infty \frac{y^n}{e^y - 1}\,dy = \frac{1}{\alpha^{n+1}}\int_0^\infty \frac{y^n e^{-y}}{1 - e^{-y}}\,dy$$

Usando series:

$$\frac{1}{(1-x)} = \sum_{n=0}^{\infty} x^n$$

$$\frac{x}{(1-x)} = \sum_{n=1}^{\infty} x^n$$

Entonces tenemos que

$$I_n = \frac{1}{\alpha^{n+1}} \sum_{k=1}^{\infty} \int_0^{\infty} y^n e^{-ky} dy$$

Sea $t = ky$:

$$I_n = \frac{1}{\alpha^{n+1}} \sum_{k=1}^{\infty} \frac{1}{k^{n+1}} \int_0^{\infty} t^n e^{-t} dt = \frac{\Gamma(n+1)}{\alpha^{n+1}} \sum_{k=1}^{\infty} \frac{1}{k^{n+1}} = \frac{\Gamma(n+1)}{\alpha^{n+1}} \zeta(n+1), \quad n > 1$$

Problema 5.9.8 Arfken–Weber Quinta edición

Las integrales de la siguiente forma apararecen el la teoría cuántica de efectos de transporte térmicos y de conductividad eléctrica.

$$I_n = \int_0^{\infty} \frac{x^n e^{\alpha x}}{(e^{\alpha x} - 1)^2} \, dx$$

Sea $y = \alpha x$:

$$I_n = \frac{1}{\alpha^{n+1}} \int_0^{\infty} \frac{y^n e^{-\alpha x}}{(1 - e^{\alpha x})^2} \, dx$$

$$\frac{d}{dx}\left(\frac{1}{(1-x)}\right) = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} (n+1) x^n$$

$$\frac{x}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1) x^{n+1} = \sum_{n=1}^{\infty} (n) x^n$$

$$I_n = \frac{1}{\alpha^{n+1}} \sum_{k=1}^{\infty} k \int_0^{\infty} y^n e^{-ky} dy$$

Sea $t = ky$:

$$I_n = \frac{1}{\alpha^{n+1}} \sum_{k=1}^{\infty} \frac{k}{k^{n+1}} \int_0^{\infty} t^n e^{-t} dt = \frac{\Gamma(n+1)}{\alpha^{n+1}} \sum_{k=1}^{\infty} \frac{1}{k^n} = \frac{\Gamma(n+1)}{\alpha^{n+1}} \zeta(n), \quad n > 1.$$

Función Zeta de Riemann

En los problemas anteriores se usó esta función que se define a continuación

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}, \quad \Re(s) > 1$$

Valores particulares:

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(3) \approx 1.2020569032$$

Para $\alpha = 1$:

$$I_3 = 3!\,\zeta(4) = 6\frac{\pi^4}{90} = \frac{\pi^4}{15}, \quad I_2 = 2!\,\zeta(3) \approx 2.4041138064$$

Otras representaciones de la función Gamma

$$\Gamma(z) = 2\int_0^{\infty} e^{-t^2} t^{2z-1} dt$$

$$\Gamma(z) = \int_0^1 \left[\ln\left(\frac{1}{t}\right)\right]^{z-1} dt$$

Forma de Weierstrass

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}$$

donde γ es la constante de Euler-Mascheroni:

$$\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n\right) \approx 0.577215\ldots$$

$$\int_0^{\infty} e^{-x} \ln x \, dx = -\gamma$$

Fórmula de multiplicación

$$\prod_{n=0}^{m-1} \Gamma\left(z + \frac{n}{m}\right) = m^{1/2-mz}(2\pi)^{(m-1)/2}\Gamma(mz)$$

Función Beta

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx, \quad m > 0, n > 0$$

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Identidades de la función Beta

$$
\begin{aligned}
&a) \quad B(a,b) = B(a+1,b) + B(a,b+1)\\
&b) \quad B(a,b) = \frac{a+b}{b} B(a,b+1)\\
&c) \quad B(a,b) = \frac{b-1}{a} B(a+1,b-1)\\
&d) \quad B(a,b)B(a+b,c) = B(b,c)B(a,b+c)
\end{aligned}
$$

Doble factorial

El doble factorial es muy importante para definir desarrollos en series

$$(2n+1)!! = 1\cdot 3\cdot 5\cdots(2n+1) = \frac{(2n+1)!}{2^n n!}$$

$$(2n)!! = 2\cdot 4\cdot 6\cdots(2n) = 2^n n!$$

Función Digamma y Poligamma

$$z! = z\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{(z+1)(z+2)\cdots(z+n)}$$

$$\ln(z!) = \lim_{n\to\infty} [\ln n! + z\ln n - \ln(z+1) - \ln(z+2) - \cdots - \ln(z+n)]$$

$$\psi(z) = \frac{d}{dz}\ln\Gamma(z+1) = \lim_{n\to\infty}\left(\ln n - \sum_{k=1}^{n}\frac{1}{z+k}\right) = -\gamma + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{z+n}\right)$$

$$\psi(0) = -\gamma$$

Poligamma:

$$\psi^{(m)}(z) = \frac{d^{m+1}}{dz^{m+1}}\ln\Gamma(z+1) = (-1)^{m+1} m! \sum_{n=1}^{\infty}\frac{1}{(z+n)^{m+1}}, \quad m = 1,2,3,\ldots$$

$$\psi^{(m)}(0) = (-1)^{m+1} m!\,\zeta(m+1)$$

Desarrollo en serie:

$$\ln\Gamma(z+1) = \sum_{n=1}^{\infty}\frac{z^n}{n!}\psi^{(n-1)}(0) = -\gamma z + \sum_{n=2}^{\infty}(-1)^n \frac{z^n}{n}\zeta(n), \quad |z| < 1$$

Funciones Gamma y Beta incompletas

Gamma incompleta:

$$\gamma(a,x)=\int_0^x e^{-t}t^{a-1}dt,\quad \Gamma(a,x)=\int_x^\infty e^{-t}t^{a-1}dt,\quad \Gamma(a)=\gamma(a,x)+\Gamma(a,x)$$

Beta incompleta:

$$B(p,q,x)=\int_0^x t^{p-1}(1-t)^{q-1}dt,\quad 0\le x\le 1, p>0, q>0$$

Integrales varias

$$\begin{aligned}
\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta &= \frac{1}{2}B(m,n)=\frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)},\quad m>0,n>0\\
\int_0^\infty \frac{x^{p-1}}{1+x}dx &= \Gamma(p)\Gamma(1-p)=\frac{\pi}{\sin(p\pi)},\quad 0<p<1\\
\int_0^\infty x^m e^{-ax^n}dx &= \frac{1}{na^{(m+1)/n}}\Gamma\left(\frac{m+1}{n}\right),\quad m>0,n>0,a>0\\
\int_0^1 x^m(\ln x)^n dx &= \frac{(-1)^n n!}{(m+1)^{n+1}},\quad n>0,m>-1\\
\int_0^\infty e^{-\alpha x^2}\cos(\beta x)\,dx &= \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}e^{-\beta^2/(4\alpha)}\\
\int_0^\infty \frac{\cos x}{x^p}dx &= \frac{\pi}{2\Gamma(p)\cos\left(\frac{p\pi}{2}\right)},\quad 0<p<1\\
\int_0^\infty \frac{\sin x}{x^p}dx &= \frac{\pi}{2\Gamma(p)\sin\left(\frac{p\pi}{2}\right)},\quad 0<p<1\\
\int_0^\infty e^{-st}\frac{dt}{\sqrt{t}} &= \sqrt{\frac{\pi}{s}},\quad s>0
\end{aligned}$$

Otras funciones especiales

Función error

$$\mathrm{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}du=1-\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-u^2}du$$

Integral exponencial

$$\mathrm{Ei}(x)=\int_x^\infty e^{-u}\frac{du}{u}$$

Integral seno y coseno

$$\mathrm{Si}(x) = \int_0^x \frac{\sin u}{u} du = \frac{\pi}{2} - \int_x^\infty \frac{\sin u}{u} du$$

$$\mathrm{Ci}(x) = \int_x^\infty \frac{\cos u}{u} du$$

Integrales de Fresnel

$$S(x) = \sqrt{\frac{2}{\pi}} \int_0^x \sin u^2 du = 1 - \sqrt{\frac{2}{\pi}} \int_x^\infty \sin u^2 du$$

$$C(x) = \sqrt{\frac{2}{\pi}} \int_0^x \cos u^2 du = 1 - \sqrt{\frac{2}{\pi}} \int_x^\infty \cos u^2 du$$

Relación con el producto infinito del seno

$$\sin(\pi x) = \pi x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right)$$

$$\cos(\pi x) = \prod_{n=1}^{\infty} \left(1 - \frac{4x^2}{(2n-1)^2}\right)$$

A continuación se demuestra la identidad de la reflexión de Euler:

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)} = \frac{1}{x} \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right)^{-1}$$

Usamos la siguiente fórmula

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}$$

$$\frac{1}{\Gamma(-z)} = -z e^{-\gamma z} \prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right) e^{z/n}$$

Multiplicamos las últimas expresiones

$$\frac{1}{\Gamma(z)\Gamma(-z)} = -z^2 \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) \left(1 - \frac{z}{n}\right)$$

$$\frac{1}{\Gamma(z)\Gamma(-z)} = -z^2 \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

Ahora empleamos la definición

$$z\sin(\pi z) = \pi z^2 \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$$

$$\frac{1}{z\sin(\pi z)} = \frac{1}{\pi z^2} \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)^{-1}$$

Entonces

$$\Gamma(z)\Gamma(-z) = -\frac{\pi}{z\sin(\pi z)}$$

$$\Gamma(1+z) = z\Gamma(z), \qquad \Gamma(1-z) = -z\Gamma(-z)$$

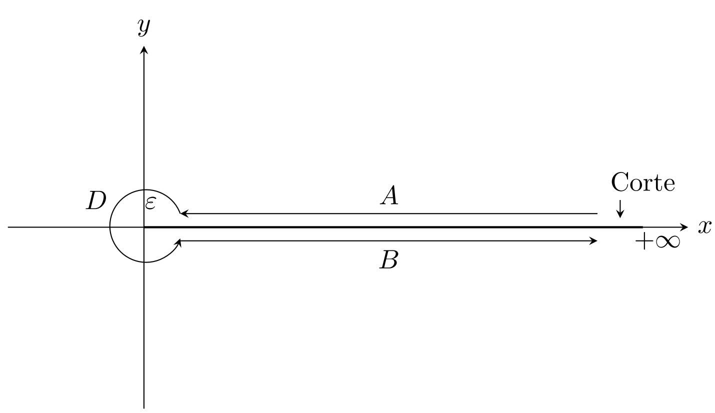
Finalmente

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

Integral de Contorno de Hankel

$$\int_C e^{-z} z^{\nu} dz = (e^{2i\pi\nu} - 1)\nu!$$

$$\int_C e^{-z} (-z)^{\nu} dz = 2i\sin(\pi\nu)\nu!$$



Problemas de funciones Gamma, Beta y Zeta de Riemann

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28 de septiembre de 2025

Problema 10.4.14 Arfeken–Weber Quinta edición

Una partícula de masa m, que se mueve en un potencial simétrico y que está bien descrito por $V(x) = A|x|^n$, tiene una energía total $1/2m(dx/dt)^2 + V(x) = E$. Encuentre dx/dt e intergre para hallar el periodo del movimiento:

$$\tau = 2\sqrt{2m}\int_0^{x_{max}} \frac{dx}{\sqrt{E - Ax^n}},$$

donde x_{max} es el punto de retorno dado por $Ax^n_{max} = E$. Demuestre que

$$\tau = \frac{2}{n}\sqrt{\frac{2\pi m}{E}}\,()^{1/n}\,\frac{\Gamma(1/n)}{\Gamma(1/n + 1/2)}.$$

Solución:

Tenemos que

$$V(x) = A|x|^n = Ax^n, \quad x > 0$$

La energía total es

$$\begin{aligned}
E &= \frac{m}{2}\left(\frac{dx}{dt}\right)^2 + V(x) \\
\sqrt{E - V(x)} &= \sqrt{\frac{m}{2}} \cdot \frac{dx}{dt} \\
dt &= \sqrt{\frac{m}{2}} \cdot \frac{dx}{\sqrt{E - V(x)}}
\end{aligned}$$

El periodo será

$$\tau = 2\sqrt{2m}\int_0^{x_{\text{máx}}} \frac{dx}{\sqrt{E - V(x)}},$$

donde $Ax^n_{\text{máx}} = E$, $x_{\text{máx}}$ es el punto de retorno.
Sean $\xi = x^n$ y $\alpha = 2\sqrt{2m}$
Así

$$d\xi = nx^{n-1}dx \Rightarrow dx = \frac{d\xi}{n\xi^{(n-1)/n}} = \frac{d\xi}{n\xi^{1-1/n}}, \qquad x = \xi^{1/n}, \qquad \xi_0 = x^n_{\text{máx}}$$

Entonces

$$\begin{aligned} \tau &= \frac{\alpha}{n}\int_0^{\xi_0} \frac{1}{\sqrt{E - A\xi}} \frac{d\xi}{\xi^{(n-1)/n}} \\ &= \frac{\alpha}{n}\int_0^{\xi_0} \frac{\xi^{-1/2}\xi^{1/n-1}}{\sqrt{A}\sqrt{\gamma/\xi - 1}} d\xi \end{aligned}$$

donde

$$\gamma = \frac{E}{A} = x^n_{\text{máx}} = \xi_0$$

Sea

$$u = \frac{\gamma}{\xi} - 1$$

Así

$$1 + u = \frac{\gamma}{\xi}, \qquad \xi = \frac{\gamma}{1+u}, \qquad d\xi = -\frac{\gamma}{(1+u)^2}du$$

Entonces

$$\begin{aligned} \tau &= -\frac{\alpha}{n}\frac{1}{\sqrt{A}}\int_\infty^0 \frac{1}{\sqrt{u}}\left(\frac{\gamma}{1+u}\right)^{1/n-3/2} \frac{\gamma}{(1+u)^2}du \\ &= \frac{\alpha}{n}\frac{1}{\sqrt{A}}\gamma^{1/n-1/2}\int_0^\infty \frac{du}{u^{1/2}(1+u)^{1/n+1/2}} \end{aligned}$$

Ahora empleamos la fórmula de la función Beta

$$\int_0^\infty \frac{u^m du}{(1+u)^{m+k+2}} = B(m+1, k+1), \qquad B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

Por comaparación tenemos que

$$m = -1/2, \qquad m + \frac{1}{4} + 2 + k = \frac{1}{n} + \frac{1}{2} \Rightarrow k = \frac{1}{n} - 1, \qquad m + 1 = \frac{1}{2}, \qquad k + 1 = \frac{1}{n}$$

Entonces

$$\int_0^\infty \frac{du}{u^{1/2}(1+u)^{1/n+1/2}} = B\left(\frac{1}{2}, \frac{1}{n}\right) = \frac{\Gamma(1/2)\Gamma(1/n)}{\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)}, \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Finalmente

$$\tau = \frac{2}{n}\sqrt{\frac{2\pi m}{E}}\left(\frac{E}{A}\right)^{1/n}\frac{\Gamma(1/n)}{\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)}$$

Problema 10.4.7 Arfken–Weber

Demuestre que la integral de Dirichlet

$$I = \iint x^p y^q dA = \frac{p!q!}{(p+q+2)!} = \frac{B(p+1,q+1)}{p+q+2}$$

donde el rango de integración es un triángulo encerrado por el eje x, eje y y la recta $x+y=1$.

Solución:

Integremos el área del triángulo

$$\begin{aligned}
I &= \int_0^1 x^p \left(\int_0^{1-x} y^q dy\right) dx \\
&= \int_0^1 x^p \left.\frac{y^{q+1}}{q+1}\right|_0^{1-x} dx \\
&= \int_0^1 \frac{x^p(1-x)^{q+1}}{q+1} dx \\
&= \frac{B(p+1,q+2)}{q+1} \\
&= \frac{p!(q+1)!}{(q+1)(p+q+2)!} = \frac{p!q!}{(p+q+2)!} \\
&= \frac{p!q!}{(p+q+2)(p+q+1)!}
\end{aligned}$$

Ahora empleamos la siguiente propiedad de las funciones Beta:

$$B(a,b) = \frac{(b-1)}{a}B(a+1,b-1) \Rightarrow B(p,q+1) = \frac{q}{p}B(p+1,q)$$

Entonces

$$I = \frac{p!q!}{(p+q+2)(p+q+1)!} = \frac{B(p+1,q+1)}{p+q+2}$$

Problema 33, página 262, Whittaker–Watson Cuarta edición

Por medio de la sustitucion $\cos\theta = 1 - 2\tan(\phi/2)$, demuestre que

$$I = \int_0^{\pi} \frac{d\theta}{(3-\cos\theta)^{1/2}} = \frac{1}{4\sqrt{\pi}} \left[\Gamma\left(\frac{1}{4}\right)\right]^2.$$

Solución:

Sea

$$\cos\theta = 1 - 2\tan\left(\frac{\phi}{2}\right)$$

Tenemos que

$$\begin{aligned}
-\sin\theta d\theta &= -\sec^2\left(\frac{\phi}{2}\right) d\varphi \\
1-\sin^2\theta &= \left(1 - 2\tan\left(\frac{\phi}{2}\right)\right)^2 = \cos^2\theta \\
&= 1 - 4\tan\left(\frac{\phi}{2}\right) + 4\tan^2\left(\frac{\phi}{2}\right) \\
\sin^2\theta &= 4\tan\left(\frac{\phi}{2}\right)\left(1 - \tan\left(\frac{\phi}{2}\right)\right) \\
\tan^2\theta + 1 &= \sec^2\theta \\
1-\sin^2\theta &= 4\sec^2\left(\frac{\phi}{2}\right) - 4\tan\left(\frac{\phi}{2}\right) - 3 \\
3-\cos\theta &= 2\left(1 + \tan\left(\frac{\phi}{2}\right)\right)
\end{aligned}$$

Sea

$$t = \tan\left(\frac{\phi}{2}\right)$$

Tenemos que

$$
\begin{aligned}
\sin\theta &= 2\sqrt{t(1-t)} \\
\cos\theta &= 1-2t \\
3-\cos\theta &= 2(1+t) \\
-\sin\theta d\theta &= -2dt \\
d\theta &= \frac{dt}{\sqrt{t(1-t)}} \\
dt &= \frac{1}{2}\sec^2\left(\frac{\phi}{2}\right)d\phi
\end{aligned}
$$

Los valores limites de la integral son

$$
\begin{aligned}
\tan\left(\frac{\phi}{2}\right) &= 0 \Rightarrow \phi = 0 \\
\tan\left(\frac{\phi}{2}\right) &= 1 \Rightarrow \sin\left(\frac{\phi}{2}\right) = \cos\left(\frac{\phi}{2}\right) \\
\phi &= \frac{\pi}{2} \Rightarrow \sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}
\end{aligned}
$$

Entonces

$$
\begin{aligned}
I &= \int_0^1 \frac{dt}{(t(1-t))^{1/2}}\frac{1}{\sqrt{2(1+t)}} \\
&= \frac{1}{\sqrt{2}}\int_0^1 t^{-1/2}(1-t^2)^{-1/2}dt = \frac{1}{2\sqrt{2}}B(m+1,n+1)
\end{aligned}
$$

Determinamos los valores de m y n

$$
\begin{aligned}
2m+1 &= -\frac{1}{2} \Rightarrow m = -\frac{3}{4} \Rightarrow m+1 = \frac{1}{4} \\
n &= -\frac{1}{2} \Rightarrow n+1 = \frac{1}{2}
\end{aligned}
$$

Tenemos que

$$
B(m+1,n+1) = \frac{\Gamma(1/4)\Gamma(1/2)}{\Gamma(3/4)} = \frac{\sqrt{\pi}\,\Gamma(1/4)}{\Gamma(3/4)}
$$

Ahora

$$
\Gamma\left(\frac{3}{4}\right) = -\frac{1}{4}\Gamma\left(-\frac{1}{4}\right)
$$

Usamos la expresión

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

Tenemos que

$$\begin{aligned}
\Gamma(-1/4)\Gamma(1+1/4) &= \Gamma(-1/4)\Gamma(5/4) = -\frac{\pi}{\sin(\pi/4)} = \frac{-2\pi}{\sqrt{2}} \\
-\frac{1}{4}\Gamma(-1/4)\Gamma(5/4) &= \frac{\pi}{2\sqrt{2}} \\
\Gamma(3/4)\Gamma(5/4) &= \frac{\pi}{2\sqrt{2}} \\
\Gamma(5/4) &= \frac{1}{4}\Gamma(1/4) \\
\Gamma(3/4) &= \frac{2\pi}{\sqrt{2}\Gamma(1/4)} = \frac{2\pi}{\sqrt{2}\Gamma(1/4)}
\end{aligned}$$

Entonces

$$B(1/4, 1/2) = \frac{\sqrt{2\pi}}{2\pi}\left[\Gamma\left(\frac{1}{4}\right)\right]^2 = \frac{1}{\sqrt{2\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2$$

Finalmente

$$I = \frac{1}{2\sqrt{2}}B(1/4, 1/2) = \frac{1}{4\sqrt{\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2.$$

Problema 10.2.4 Arfken–Weber Quinta edición

Demuestre que

$$\frac{1}{2}\ln\left(\frac{\pi z}{\sin(\pi z)}\right) = \sum_{n=1}^{\infty}\frac{\zeta(2n)}{2n}z^{2n}, \qquad |z| < 1$$

Solución:

Empleamos la siguiente igualdad

$$\begin{aligned}
F^{(m)}(z) &= \frac{d^{m+1}}{dz^{m+1}}\ln\Gamma(z+1) = \frac{d^{m+1}}{dz^{m+1}}\ln z! \\
&= (-1)^{m+1}m!\sum_{n=1}^{\infty}\frac{1}{(z+n)^{m+1}}, \qquad m = 1, 2, 3, \cdots \\
F^{(m)}(0) &= (-1)^{m+1}m!\,\zeta(m+1).
\end{aligned}$$

Ahora usamos otra expresión

$$
\begin{aligned}
\ln(z!) &= \sum_{n=1}^{\infty} \frac{z^n}{n!} F^{(n-1)}(0) = -\gamma z + \sum_{n=2}^{\infty} (-1)^n \frac{z^n}{n} \zeta(n), \quad |z| < 1 \\
\ln((-z)!) &= \gamma z + \sum_{n=2}^{\infty} (-1)^{2n} \frac{z^n}{n} \zeta(n) \\
\ln(z!) + \ln((-z)!) &= \sum_{n=2}^{\infty} \frac{z^n}{n} (1 + (-1)^n) \zeta(n) \\
&= 2 \sum_{n=1}^{\infty} \frac{z^{2n}}{2n} \zeta(2n) \\
&= \ln(z!(-z)!) = \ln(\Gamma(1+z)\Gamma(1-z)) \\
&= \ln\left(\frac{\pi z}{\sin(\pi z)}\right) \\
\Rightarrow \frac{1}{2} \ln\left(\frac{\pi z}{\sin(\pi z)}\right) &= \sum_{n=1}^{\infty} \frac{\zeta(2n)}{2n} z^{2n}
\end{aligned}
$$

Problema 10.4.16 Arfken–Weber Quinta edición

Muestre que

$$
I = \int_0^{\infty} \frac{\sinh^{\alpha} x}{\cosh^{\beta} x} dx = \frac{1}{2} B(\gamma, \delta), \quad -1 < \alpha < \beta
$$

donde $\gamma = (1+\alpha)/2$, $\delta = (\beta - \alpha)/2$

Solución:

Sea

$$
u = \sinh^2 x \Rightarrow \sinh x = \sqrt{u}
$$

Así

$$
du = 2 \sinh x \cosh x dx
$$

Además

$$
\begin{aligned}
\cosh^2 x - \sinh^2 x &= 1 \\
\cosh^2 x &= 1 + \sinh^2 x = 1 + u \\
\cosh x &= \sqrt{1+u}
\end{aligned}
$$

Entonces

$$\begin{aligned} I &= \int_0^\infty u^{\alpha/2}(1+u)^{-\beta/2}\frac{du}{2\sqrt{u}\sqrt{1+u}} \\ &= \frac{1}{2}\int_0^\infty u^{(\alpha-1)/2}(1+u)^{-(\beta+1)/2}du \end{aligned}$$

Ahora

$$B(m+1,n+1)=\int_0^\infty \frac{u^m}{(1+u)^{m+n+2}}du$$

Procedemos a determinar los exponentes

$$\begin{aligned} m &= \frac{\alpha-1}{2} \\ m+1 &= \frac{\alpha+1}{2} \\ m+n+2 = \frac{\beta+1}{2} \\ m+1+n+1 &= \frac{\alpha+1}{2}+n+1=\frac{\beta+1}{2} \\ \frac{\alpha+1}{2}+n &= \frac{\beta-1}{2} \\ n &= \frac{\beta-1}{2}-\frac{\alpha+1}{2}=\frac{1}{2}(\beta-1-\alpha-1)=\frac{1}{2}(\beta-\alpha-2) \\ n+1 &= \frac{1}{2}(\beta-\alpha) \end{aligned}$$

Entonces

$$\begin{aligned} I &= \frac{1}{2}B\left(\frac{1}{2}(\alpha+1),\frac{1}{2}(\beta-\alpha)\right) \\ &= \frac{1}{2}B\left(\frac{\alpha-1}{2}+1,\frac{1}{2}(\beta-\alpha-2)+1\right)]] \\ &= \frac{1}{2}\frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)} \\ &= \frac{1}{2}\frac{m!n!}{(m+n+1)!}=\frac{((\alpha-1)/2)!((\beta-\alpha-2)/2)!}{((\beta-1)/2)!} \end{aligned}$$

Problema 9.10, página 226, Spiegel, Matemáticas Superiores

Una partícula es atraída hacia un punto fijo O con una fuerza que es inversamente proporcional a su distancia al punto O. Si la partícula sale del reposo, encontrar el tiempo que necesita para llegar a O.

Solución:

Las condiciones iniciales son

$$t = 0, \qquad r = a > 0, \qquad v_0 = 0$$

La fuerza está dada por

$$F = -\frac{k}{r} \Rightarrow \partial_r V = \frac{k}{r}$$

Ahora

$$F = ma = m\frac{d^2 r}{dt^2} = -\frac{k}{r} = -\partial_r V$$

Sea

$$v = \frac{dr}{dt}$$

Entonces

$$\begin{aligned} m\frac{dv}{dt} &= -\frac{k}{r} \\ m\frac{dv}{dt} &= m\frac{dv}{dr}\frac{dr}{dt} = mv\frac{dv}{dr} \\ \frac{d}{dr}\left(\frac{m}{2}v^2\right) &= -\frac{k}{r} \\ d\left(\frac{m}{2}v^2\right) &= -\frac{k}{r}dr = -kd(\ln r) \\ \frac{m}{2}v^2 &= -k\ln r + c_0 \end{aligned}$$

Por las condiciones iniciales encontramos c_0

$$v_0 = 0, \quad r = a \Rightarrow 0 = -k\ln a + c_0$$

Por lo tanto

$$\begin{aligned} \frac{m}{2}v^2 &= k\ln\left(\frac{a}{r}\right) \\ \Rightarrow v &= \frac{dr}{dt} = \pm\sqrt{\frac{2k}{m}\ln\left(\frac{a}{r}\right)} \end{aligned}$$

Escogemos el signo negativo (hacia O):

$$\begin{aligned} v = \frac{dr}{dt} &= -\sqrt{\frac{2k}{m}\ln\left(\frac{a}{r}\right)} \\ \frac{dr}{\sqrt{\ln\left(\frac{a}{r}\right)}} &= -\sqrt{\frac{2k}{m}}dt \\ \sqrt{\frac{2k}{m}}\int_0^T dt &= \sqrt{\frac{2k}{m}}T = \int_0^a \frac{dr}{\sqrt{\ln\left(\frac{a}{r}\right)}} \end{aligned}$$

Sea

$$u = \ln\left(\frac{a}{r}\right) \Rightarrow e^u = \frac{a}{r} \Rightarrow r = ae^{-u}$$

Ahora

$$du = -\frac{1}{r}dr \Rightarrow dr = -rdu = -ae^{-u}du$$

Cuando $r = 0 \Rightarrow u = \infty$, $r = a \Rightarrow u = 0$

$$\begin{aligned} T &= \sqrt{\frac{m}{2k}}\int_0^a \frac{dr}{\sqrt{\ln\left(\frac{a}{r}\right)}} \\ &= \sqrt{\frac{m}{2k}}\int_\infty^0 \frac{-ae^{-u}du}{\sqrt{u}} \\ &= a\sqrt{\frac{m}{2k}}\int_0^\infty u^{-1/2}e^{-u}du \\ &= a\sqrt{\frac{m}{2k}}\Gamma\left(\frac{1}{2}\right) \\ &= a\sqrt{\frac{\pi m}{2k}} \end{aligned}$$

Problema 10.1.18 Arfken–Weber Quinta edición

De alguna de las definiciones del factorial o la función Gamma muestre que

$$|(ix)!|^2 = \frac{\pi x}{\sinh(\pi x)}$$

Solución:

Partimos de

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \Rightarrow \Gamma(1+z)\Gamma(1-z) = \frac{\pi z}{\sin(\pi z)}$$

Sea $z = ix$

$$\begin{aligned}
\Gamma(ix)\Gamma(1-ix) &= \frac{\pi}{\sin(\pi ix)} \\
\Gamma(1+ix) &= (ix)! \\
\Gamma(1-ix) &= (-ix)! \\
(ix)!(-ix)! &= \frac{\pi(ix)}{\sin(\pi ix)} = \frac{\pi x}{\frac{\sin(i\pi x)}{i}} = \frac{\pi x}{\sinh(\pi x)} \\
|(ix)!|^2 &= (ix)![(ix)!]^* = (ix)!(-ix)! = \frac{\pi x}{\sinh(\pi x)}
\end{aligned}$$