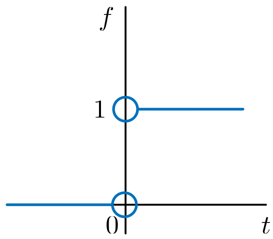
Fourier transform

Mathematical review

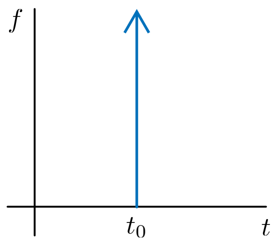
Cauchy's principal value (CPV): for a function f $:]-\infty,\infty[\rightarrow\mathbb{R}$ the limiting value CPV $\int_{-\infty}^{\infty} f(x)dx = \lim_{a\to\infty}\int_{-a}^{a} f(x)dx$, if it exists, is called the Cauchy's principal value.



Heaviside's function: piecewise function $H:\mathbb{R}\rightarrow\mathbb{C}$, $H(t)=\begin{cases}1, & t>0\\ 0, & t<0\end{cases}$. There are different conventions for $H(0)$ $(=1, \quad =1/2, \quad =\nexists)$.

Distributions or generalized functions: they extend the concept of function such that certain derivatives or integrals exist despite the function not being continuous and infinitely differentiable. Distributions are often defined in terms of a limit involving classical functions.

Dirac delta: distribution $\delta:\mathbb{R}\rightarrow\mathbb{R}$ defined by $\delta(t-t_0)=\begin{cases}0, & t\neq 0\\ \infty, & t=0\end{cases}$ such that given a continuous function g, $\int_{-\infty}^{\infty} g(t)\delta(t-t_0)dt = g(t_0)$.



The Dirac delta can be defined by using several *representations*. Here we review two of them:

- The Dirac delta is the derivative of the Heaviside function: $\delta(t-t_0)=\frac{d}{dt}H(t-t_0)$. This can be shown by approximating the Heaviside function by continous functions where the 0 and the 1 are connected by a line within an ϵ around t_0. As $\epsilon\rightarrow 0$ the line becomes more vertical and therefore the derivative (slope) becomes infinite at only one point (just like the Dirac delta).
- The Dirac delta can be defined as $\delta(t-t_0)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(t-t_0)x}dx$. Conceptually, one can think of the integral $\int_{-L}^{L} e^{ixt}dx = \left.\frac{\exp(ixt)}{it}\right|_{-L}^{L} = \frac{2\sin(Lt)}{t}$. Functions similar to $\sin(t)/t$ have a spike at $t=0$ and vanish for big t, similar to a Dirac delta.
- Properties:
  - Scaling: For $a\in\mathbb{R}\backslash\{0\}$, $\delta(ax)=\frac{\delta(x)}{|a|}$ because $\int_{-\infty}^{\infty}\delta(ax)dx=\int_{-\infty}^{\infty}\delta(u)dx/|a| = 1/|a|$ (the absolute value is necessary to keep the integration limits).
  - From the scaling property $\Longrightarrow$ $\delta(-x)=\delta(x)$
  - Generalization with a function: $\delta(f(x))=\sum_k \frac{1}{|f'(x_k)|}\delta(x-x_k)$, where the x_k are all the roots of f. Example: $\delta(ax-b)=|a|^{-1}\delta(x-b/a)$ for $a,b\in\mathbb{R}, a\neq 0$.
  - In n dimensions, $\delta(a\mathbf{x})=|a|^{-n}\delta(\mathbf{x})$.
  - Distribution: $x\delta(x)=0$ because $\int_{-\infty}^{\infty} x\delta(x)f(x)dx=(0)\cdot f(0)=0$. In general, $(x-a)^n\delta(x-a)=0$ for $n\in\mathbb{N}, a\in\mathbb{R}$.

Fourier transform: simplified derivation: the expansion of a function in a Fourier series requires the function to be defined within an interval and be periodic. As a result, we obtain a sum (series) of harmonic functions with discrete amplitudes. In order to remove these constraints (namely, to use a non-periodic function and continuous amplitudes), we need to use a Fourier transform.

Suppose we take a periodic function $f(t) : [-T/2, T/2] \to \mathbb{R}$. We build a Fourier series, but this time, let's define the angular frequency $\omega_k = 2\pi k/T$ (which implies $\Delta\omega = 2\pi/T$, since $\Delta k = 1$ because $k \in \mathbb{Z}$). Substituting, we have

$$f(t) = \sum_{k=\infty}^{\infty} \left[\frac{\Delta\omega}{2\pi} \int_{-T/2}^{T/2} f(t) e^{-i\omega_k t} dt \right] e^{i\omega_k t}$$

(the term inside the brackets are the coefficients c_k). Now, we do $T \to \infty \implies \Delta\omega \to 0$, which implies that the sum becomes an integral and the angular frequency becomes continuous:

$$f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left[\int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \right] e^{i\omega t}$$

The innermost integral (the one inside the brackets) is called the *Fourier transform* $F(\omega)$ of $f(t)$, and the outermost integral is called the *inverse Fourier transform*. There are different conventions on how to deal with the factor of $1/(2\pi)$: some authors distribute it equally between the two transforms as factors of $1/\sqrt{2\pi}$, others use the factor of $1/(2\pi)$ only on the inverse Fourier transform. In this document, we take the factor only on the inverse transform.

Fourier transform: definition: for a function $f : \mathbb{R} \to \mathbb{C}$, the function $F(\omega) = \text{CPV} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$ is called the Fourier transform of f, and one can use for it the notation $\mathcal{F}\{f(t)\} = F(\omega)$.

Interpretation: the Fourier transform gives out a complex function of the angular frequency. It originated from the coefficients of the Fourier series, so it can be interpreted as the (complex) amplitudes of the sum of (complex) harmonic functions $e^{i\omega t}$ that describe $f(t)$. The fact that those functions are complex means that the Fourier transform also encodes information on the phase of those harmonic functions (think of Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$).

- Trivial example: the Fourier transform of $f(t) = e^{i\omega_0 t}$ is $F(\omega) = \int_{-\infty}^{\infty} e^{i(\omega_0 - \omega)t} dt$ $= 2\pi\delta(\omega_0 - \omega) = 2\pi\delta(\omega - \omega_0)$. This means that in the case of a harmonic function in t (usually time space) we get a spike at ω_0 in ω (usually the frequency space). Using exponential functions, we can build the Fourier transform of sines and cosines (sums of Dirac delta functions, some multiplied by i).

Inverse Fourier transform: for an image function $F(\omega)$, we define the function $\tilde{f} : \mathbb{R} \to \mathbb{C}$ with $\tilde{f}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} F(\omega) d\omega$, as long as it exists for each $t \in \mathbb{R}$, as the inverse Fourier transform of F. In the case that $f(t)$ is Fourier-transformable and differentiable in portions and its transform is F, then the inverse transform of F is

$$\tilde{f}(t) = \begin{cases} f(t), & \text{if } f \text{ is continuous} \\ \frac{f(t^+) + f(t^-)}{2}, & \text{if } f \text{ is discontinuous} \end{cases}$$

In practice, one does not compute directly the inverse transformation because the integrals typically become difficult, but instead one uses the properties of the Fourier transform described below to express $F(\omega)$ in terms of known transformations.

Fourier transform: properties: Exercise: show all of them using the definition of Fourier transform

- Linearity: for all $\lambda, \mu \in \mathbb{C}$, if $F(\omega) = \mathcal{F}\{f(t)\}$ and $G(\omega) = \mathcal{F}\{g(t)\}$, then $\mathcal{F}\{\lambda f(t) + \mu g(t)\} = \lambda F(\omega) + \mu G(\omega)$.
  - Application: consider a real signal $f(t) = 3A\cos(\omega_0 t) + A\cos([2\omega_0]t)$ (for example, the sound of a typical musical instrument, where the base frequency ω_0 is "louder" than the first harmonic $2\omega_0$). Given that $\mathcal{F}\{\cos(\omega_0 t)\} = \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$, one can see that $F(\omega) = 3A\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + A\pi[\delta(\omega - 2\omega_0) + \delta(\omega + 2\omega_0)]$. This means that for a

given signal, the Fourier transform selects all the frequencies present in it (it gives out a complex spectrum).

- Conjugation: $\mathscr{F}\{\overline{f(t)}\} = \overline{F(-\omega)}$
- Scaling: $\mathscr{F}\{f(ct)\} = \frac{1}{|c|}F\left(\frac{\omega}{c}\right)$ for $c \in \mathbb{R}\backslash\{0\}$. Note that because of the properties of the Dirac delta, $\mathscr{F}\{e^{2i\omega_0 t}\} = 2\pi \cdot \frac{1}{2}\delta(\omega/2 - \omega_0) = 2\pi \cdot \delta(\omega - 2\omega_0)$.
- Displacement in time: $\mathscr{F}\{f(t-a)\} = e^{-i\omega a}F(\omega)$
- Displacement in frequency: $\mathscr{F}\{e^{i\omega' t}f(t)\} = F(\omega - \omega')$ for $\omega' \in \mathbb{R}$
- Derivative with respect to time: $\mathscr{F}\{f'(t)\} = i\omega F(\omega)$
- Derivative with respect to frequency: $\mathscr{F}\{tf(t)\} = iF'(\omega)$
- Convolution theorem („Faltungsprodukt"): We define a convolution of f, g as $(f * g)(t) = \int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau$. The Fourier transform of a convolution is $\mathscr{F}\{f * g\} = F(\omega)G(\omega)$. Another way to state this is using the inverse Fourier transform: $\mathscr{F}^{-1}\{F(\omega)G(\omega)\} = \int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau$.
- Duality: if f is continuous in t, $\mathscr{F}\{F(t)\} = 2\pi f(-\omega)$. Example: we saw above that $\mathscr{F}\{e^{i\omega_0 t}\} = \delta(\omega - \omega_0)$. Then, it's not surprising that $\mathscr{F}\{\delta(t - t_0)\} = e^{-i\omega t_0}$.
- Symmetry: if the function f is even or odd, then F is also even or odd.
- Inversion: $\mathscr{F}\{f(-t)\} = F(-\omega)$
- Parseval's identity: $\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega = \int_{-\infty}^{\infty} [f(t)]^2 dt$. Exercise: prove this relation (hint: substitute one of the $f(t)$ by the inverse Fourier transform definition).
- General Parseval's relation: $\int_{-\infty}^{\infty} F(\omega)\overline{G(\omega)}d\omega = \int_{-\infty}^{\infty} f(t)\overline{g(t)}dt$.

$f(t)$	$F(\omega)$	$f(t)$	$F(\omega)$
$\text{rect}(t) = \begin{cases} 0, & \lvert t \rvert \geq 1/2 \\ 1, & \lvert t \rvert < 1/2 \end{cases}$ $= H(t+1/2) - H(t-1/2)$	$\text{sinc}(\omega/2) = \frac{\sin(\omega/2)}{\omega/2}$	1	$2\pi\delta(\omega)$
$e^{-at}H(t)$	$\frac{1}{a+i\omega}$	e^{iat}	$2\pi\delta(\omega - a)$
e^{-at^2}	$\sqrt{\frac{\pi}{a}}e^{-\omega^2/(4a)}$	$x^n,\ n \in \mathbb{N}$	$2\pi i^n \delta^{(n)}(\omega)$, where (n) means derivative
$e^{-a\lvert t \rvert},\ a \in \mathbb{R}, a > 0$	$\frac{2a}{a^2+\omega^2}$	$1/x$	$-i\pi\,\text{sgn}(\omega)$, where sgn is the sign function

Fourier transform in *n* dimensions: the Fourier transform can be generalized to n dimensions. For this definition, we use a different normalization than in the rest of this document:

$$f(\mathbf{r}) = \frac{1}{(2\pi)^{n/2}}\int d^n k\ e^{i\mathbf{k}\cdot\mathbf{r}}F(\mathbf{k})$$
$$F(\mathbf{k}) = \frac{1}{(2\pi)^{n/2}}\int d^n x\ e^{-i\mathbf{k}\cdot\mathbf{r}}f(\mathbf{r})$$

References

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