Partial differential equations

Problem. The vertical displacement y of a string when a wave is propagating through it is described by the equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

where x is the horizontal position of the points belonging to the string, t is time and c is the propagation speed.
a) Verify that $y = f_1(x-ct)$ and $y=f_2(x+ct)$ are solutions to the wave PDE.

$$y = f_1(x \pm ct)$$

$$\frac{\partial y}{\partial x} = \frac{\partial f(x\pm ct)}{\partial (x\pm ct)} \underbrace{\frac{\partial (x\pm ct)}{\partial x}}_{=1} \qquad \frac{\partial y}{\partial t} = \frac{\partial f(x\pm ct)}{\partial (x\pm ct)} \underbrace{\frac{\partial (x\pm ct)}{\partial t}}_{=\pm c}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 f(x\pm ct)}{\partial (x\pm ct)^2} \underbrace{\frac{\partial (x\pm ct)}{\partial x}}_{=1} \qquad \frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 f(x\pm ct)}{\partial (x\pm ct)^2}\cdot c^2$$

$$\Rightarrow \frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

$$\Leftrightarrow \frac{\partial^2 f(x\pm ct)}{\partial (x\pm ct)^2} = \frac{1}{c^2}\cdot c^2 \frac{\partial^2 f(x\pm ct)}{\partial (x\pm ct)^2} \quad \checkmark$$

b) Express f_1, f_2 in terms of the initial conditions $y = y_0(x)$, $\frac{\partial y}{\partial t} = V_0(x)$ both at $t=0$.

The general solution is

$$y = f_1(x-ct) + f_2(x+ct)$$

therefore, at $t=0$ we have

$$y_0(x) = f_1(x) + f_2(x) \qquad (1)$$

$$V_0(x) = -cf_1'(x) + cf_2'(x) \qquad (2)$$

$$\frac{d}{dx}(1): \quad y_0' = f_1' + f_2' \quad \Big| \longrightarrow (2)$$

$$V_0 = -c(y_0' - f_2') + cf_2'$$

$$\Rightarrow 2cf_2' = V_0 + cy_0'$$

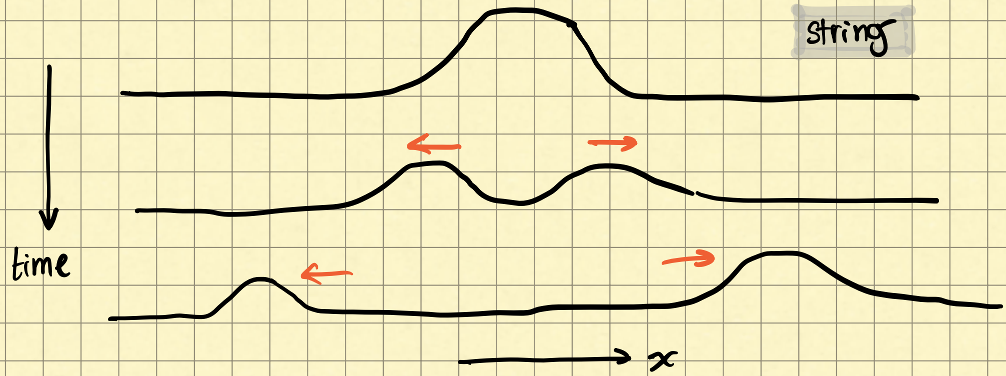
$$\Rightarrow \begin{cases} f_2 = \frac{1}{2}\left[y_0 + \frac{1}{c}\int_{x_1}^{x} V_0(x')\,dx'\right] \\ f_1 = \frac{1}{2}\left[y_0 - \frac{1}{c}\int_{x_1}^{x} V_0(x')\,dx'\right] \end{cases}$$

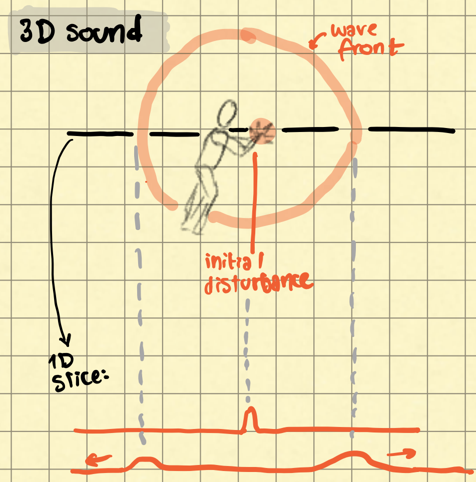
$$\therefore\ y(x,t) = \frac{1}{2}\left[y_0(x-ct) + y_0(x+ct) + \frac{1}{c}\int_{x-ct}^{x+ct} V_0(x')\,dx'\right]$$

c) Find the general form of the wave solution for the initial condition $V_0(x) = 0$
(this is an example of an initial condition onto a hyperbolic PDE in Cartesian coordinates)

$$y(x,t) = \frac{1}{2}\left[y_0(x-ct) + y_0(x+ct)\right]$$

Interpretation: for a string with an initial disturbance $y_0(x)$ (the actual shape doesn't matter), 1/2 of the produced wave propagates forwards (+x) and 1/2 propagates backwards (−x).





Notice that to solve this second-order PDE we need two initial conditions (velocity and position, although the initial position is a free function in this general problem, we still need it for a concrete problem). Note that the fact that the functions are expressed as a function of $x \pm ct$ means that this problem can be solved by the method of characteristics, with the characteristic speed c

Problem. A string of variable density $\rho(x)$ held at a variable tension $F(x)$ satisfies the PDE

$$\rho(x)\, \partial_t^2 y = \partial_x \left(F(x)\, \partial_x y \right)$$

show that this equation is separable. This is an example of a hyperbolic PDE in Cartesian coordinates.

let's assume $y(x,t) = X(x)\,T(t)$.

$$\Rightarrow \quad \begin{aligned} \partial_x y &= X'(x)\,T(t) \\ \partial_x^2 y &= X''(x)\,T(t) \\ \partial_t y &= X(x)\,T'(t) \\ \partial_t^2 y &= X(x)\,T''(t) \end{aligned}$$

$$\rho(x)\,X(x)\,T''(t) = F'(x)\,X'(x)\,T(t) + F(x)\,X''(x)\,T(t) \quad \Big| \times \frac{1}{X(x)\,T(t)\,\rho(x)}$$

$$\frac{T''(t)}{T(t)} = \frac{F'(x)\,X'(x)}{X(x)\,\rho(x)} + \frac{F(x)\,X''(x)}{X(x)\,\rho(x)} \overset{!}{=} \text{const} \quad \checkmark$$

Problem: solve the wave equation using separable variables for a constant propagation speed for a string held fixed $(y=0)$ at $x=0$, $x=L$. This is a hyperbolic PDE in Cartesian coordinates with Dirichlet boundary conditions.

$$c^2\, \partial_x^2 y = \partial_t^2 y$$

Assuming $y(x,t) = X(x)\,T(t)$

$$c^2 X'' T = X T'' \quad \Big| \times \frac{1}{XT}$$

$$\Rightarrow \quad c^2 \frac{X''}{X} = \frac{T''}{T} = -\lambda \quad : \text{const. } \underline{\text{to be determined.}}$$

$$X'' + \frac{\lambda}{c^2} X = 0 \qquad\qquad T'' + \lambda T = 0$$

$\underline{\underline{X}}$:

$$X = A\sin\left(\frac{\sqrt{\lambda}}{c}x + \phi\right)$$

boundary cond.: at $x=0$, $X \overset{!}{=} 0$

$$X(0) = A\sin\left(\frac{\sqrt{\lambda}}{c}\cdot 0 + \phi\right) = A\sin\phi \overset{!}{=} 0 \Rightarrow \phi = 0.$$

boundary cond.: at $x = L$, $X \overset{!}{=} 0$

$$X(L) = A\sin\left(\frac{\sqrt{\lambda}}{c}L\right) \overset{!}{=} 0$$

$$\Leftrightarrow \frac{\sqrt{\lambda}}{c}L = n\pi \quad \Rightarrow \lambda = \left(\frac{n\pi c}{L}\right)^2$$

$$\therefore X(x) = A\sin\left(\frac{n\pi x}{L}\right)$$

$\underline{\underline{T}}$:

$$T = B\sin\left(\frac{n\pi c t}{L} - \theta\right)$$

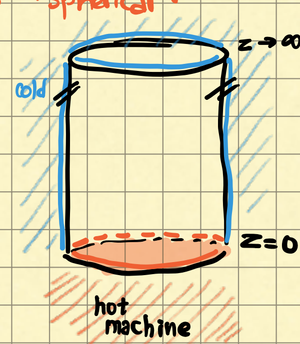
$$\therefore y = XT = \underline{y_{max}}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi c t}{L} - \underline{\theta}\right)$$

determined by other initial conditions (initial shape, initial velocity)

Problem: consider a cylindrical pipe of radius a which allows a hot machine located at the bottom of the pipe $(z=0,\ u=100)$ to diffuse heat through it and cool down to the outside temperature of $u=0$. The temperature within the pipe can be modeled using the Laplace equation

$$\nabla^2 u(r,z) = 0$$

Warning: here, r is $r_{cylindrical}$ and <u>not</u> $r_{spherical}$.



and the boundary conditions $u(r=a)=0;\ u(z=0)=100;\ u(z\to\infty)=0$
Find the solution for the temperature inside the pipe in the stationary state (no time dependence).

This is an example of an elliptical PDE in cylindrical coordinates under Dirichlet boundary conditions.

in cylindrical coord., the Laplace operator is

$$\nabla^2 u(r,z) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2}$$

Let $u(r,z) = R(r)Z(z)$. Then,

$$\frac{Z}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + RZ'' = 0 \quad \Big| \times \frac{1}{RZ}$$

$$\frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\frac{Z''}{Z} = -k^2 : \text{const}$$

$\underline{\underline{Z}}$: $Z'' = k^2 Z \quad \Rightarrow \quad Z(z) \in \{e^{-kz}\}$ because $Z(z\to\infty) \longrightarrow 0$.

(the solution e^{+kz} doesn't satisfy the bound. cond.)

$\underline{R}$: $\frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -k^2 \quad \Big| \cdot r^2 R$

$\Rightarrow r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + k^2 r^2 R = 0$: Bessel ODE (change of var. $x = kr$)

$\Rightarrow R(r) \in \{ J_0(kr), \cancel{Y_0(kr)}^{\,0} \}$ singularity at $r=0$, but $u \in \mathbb{R}$ at $r=0$

$\in \{ J_0(kr) \}$

$\therefore u(r,z) \in \{ J_0(kr) e^{-kz} \}$

• boundary cond.: $u(r=a, z) = 0 \Rightarrow R(a) = 0$

$\Rightarrow J_0(ka) = 0$

$\Rightarrow ka = \underbrace{c_{0m}}_{\text{zero of Bessel function}}$ where $m \in \mathbb{N}$.

scipy.jn_zeros(0, m)

$$\Rightarrow u(r,z) = \sum_{m=1}^{\infty} b_m \, J_0\!\left(c_{0m}\frac{r}{a}\right) e^{-c_{0m} z/a}$$

• boundary cond.: $u(r<a, z=0) = 100$

$$u(r<a, 0) = \underbrace{\sum_{m=1}^{\infty} b_m J_0\!\left(c_{0m}\frac{r}{a}\right) \cdot 1}_{\text{series}} \overset{!}{=} 100$$

$$\sum_{m=1}^{\infty} b_m \langle J_0(c_{0\ell}\, r/a), J_0(c_{0m}\, r/a) \rangle = \langle J_0(c_{0\ell}\, r/a), 100 \rangle$$

Let $x \equiv r/a$.

$$\sum_{m=1}^{\infty} b_m \int_0^1 dx \, \underbrace{x\, J_0(x c_{0\ell}) J_0(x c_{0m})}_{\text{orthogonality}} = \int_0^1 dx \, \underbrace{100\, x\, J_0(c_{0\ell} x)}_{\text{recurrence}}$$

recurrence: $y J_0(y) = (y J_1(y))'$

$$\sum_{m=1}^{\infty} b_m \, \delta_{m\ell} \cdot \frac{1}{2} \left[J_1(c_{0m}) \right]^2 = \frac{100\, J_1(c_{0\ell})}{c_{0\ell}}$$

$$\Rightarrow b_\ell = \frac{200}{c_{0\ell}\, J_1(c_{0\ell})}$$

$$\therefore u(r,z) = 200 \sum_{m=1}^{\infty} \frac{1}{c_{0m} J_1(c_{0\ell})} \cdot J_0\!\left(c_{0m}\frac{r}{a}\right) \cdot e^{-c_{0m} z/a}$$

Problem: the electrical potential can be computed with the Laplace equation $\nabla^2 u(r,\theta,\varphi) = 0$ in spherical coordinates

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right] u(r,\theta,\varphi) = 0$$

Find all the base functions that form the general solution and compare against the simplest case of a point charge $\left(u \propto 1/r\right)$

This is an example of an elliptical PDE in spherical coordinates.

Ansatz: $u(r,\theta,\varphi) = R(r)\,\Theta(\theta)\,\Phi(\varphi)$

$$\frac{\Theta\Phi}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{R\Phi}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{R\Theta}{r^2\sin^2\theta}\frac{d^2\Phi}{d\varphi^2} = 0 \quad \Big| \cdot \frac{r^2}{R\Theta\Phi}$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{1}{\Phi\sin^2\theta}\frac{d^2\Phi}{d\varphi^2} = \ell(\ell+1): \text{ const (to be det.)}$$

<u>R:</u>

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \ell(\ell+1)$$

Let $R = U/r \Rightarrow r^2 U'' = \ell(\ell+1)\,U \longrightarrow U(r) \in \{r^{-\ell}, r^{\ell+1}\}$

<u>Θ, Φ:</u>

$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi\sin^2\theta}\frac{d^2\Phi}{d\varphi^2} = -\ell(\ell+1) \quad \Big| \cdot \sin^2\theta$$

$$\Rightarrow \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \ell(\ell+1)\sin^2\theta = -\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = m^2 : \text{ const (to be det.)}$$

<u>Φ:</u>

$$\Phi'' + m^2\Phi = 0 \longrightarrow \Phi(\varphi) \in \{e^{\pm im\varphi}\}$$

<u>Θ:</u>

$$\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left(\ell(\ell+1)\sin^2\theta + m^2\right)\Theta = 0 \quad \Big| \cdot \frac{1}{\sin^2\theta}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left(\ell(\ell+1) + \frac{m^2}{\sin^2\theta}\right)\Theta = 0$$

with the substitution $x = \cos\theta$ one can show this is the Associate Legendre ODE

$$\longrightarrow \Theta(\theta) \in \{P_\ell^m(\cos\theta)\}$$

The base functions $\{P_\ell^m(\cos\theta)\,e^{\pm im\varphi}\}$ form the *spherical harmonics*.

The normalized, conventional spherical harmonics are written as

$$Y_{\ell m}(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\,P_\ell^m(\cos\theta)\,e^{im\varphi}$$

with $Y_{\ell\,-m} = (-1)^m\,\overline{Y}_{\ell m}$, and normalization

$$\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, d\theta \; \overline{Y}_{\ell'm'}(\theta,\varphi)\, Y_{\ell m}(\theta,\varphi) = \delta_{\ell'\ell}\, \delta_{m'm}$$

A function can be expanded in a series as

$$f(\theta,\varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_{\ell m}\, Y_{\ell m}(\theta,\varphi)$$

with the coefficients $f_{\ell m} = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, d\theta \; \overline{Y}_{\ell m}\, f(\theta,\varphi)$

which imply the relation

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \overline{Y}_{\ell m}(\theta',\varphi')\, Y_{\ell m}(\theta,\varphi) = \delta(\cos\theta - \cos\theta')\, \delta(\varphi-\varphi')$$

when $f(\theta,\varphi) = \delta(\cos\theta - \cos\theta')\, \delta(\varphi - \varphi')$ is used.

Problem: consider the diffusion equation

$$\frac{\partial T}{\partial t} - \beta \frac{\partial^2 T}{\partial z^2} = 0$$

describing the time-dependent heat transfer within a medium, where β is a constant.
Solve the equation with the boundary conditions

(Dirichlet boundary condition) $T(z=0) = 0$ which means constant temperature of 0 at $z=0$

(Neumann boundary condition) $\left.\frac{dT}{dz}\right|_{z=a} = -\alpha T\Big|_{z=a}$ which means black-body radiation at $z=a$

This is an example of a parabolic PDE in Cartesian coordinates under Cauchy boundary conditions.

Ansatz: $T = f(t)\, g(z)$

$$\Rightarrow \frac{1}{\beta f}\frac{df}{dt} = \frac{1}{g}\frac{d^2 g}{dz^2} \equiv -\lambda^2 \quad \text{(const. to be det.)}$$

time:

$$\frac{df}{dt} = -\lambda^2 \beta f$$

const. of integr.

$$\Rightarrow \frac{df}{f} = -\lambda^2\beta\, dt \Rightarrow \ln f/f_0 = -\lambda^2 \beta t$$

$$\Rightarrow f = f_0\, e^{-\lambda^2\beta t} \quad \text{or} \quad f(t) \in \{ e^{-\lambda^2 \beta t} \}$$

Space:

$$\frac{d^2 g}{dz^2} = -\lambda^2 g$$

$$\Rightarrow g = A\sin\lambda z + B\cos\lambda z$$

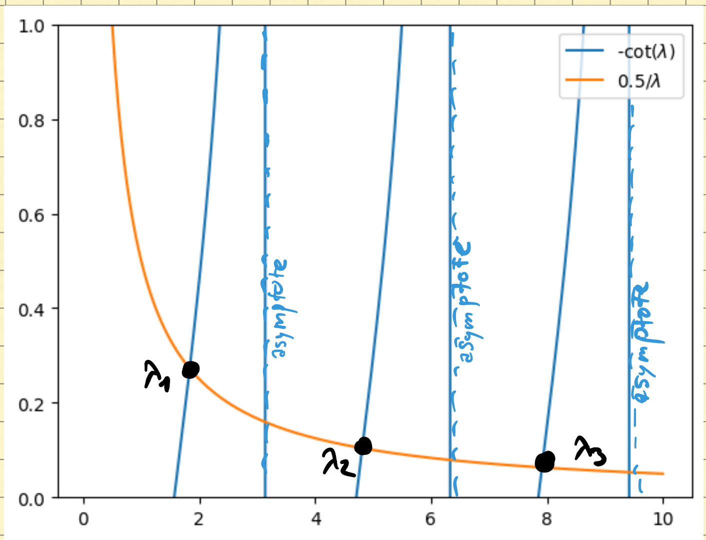
- At $z=0$, $g \overset{!}{=} 0 \Rightarrow B \overset{!}{=} 0$

$$\Rightarrow g = A\sin\lambda z$$

- At $z=a$,

$$g = A\sin\lambda a$$
$$g' = A\lambda\cos\lambda a$$
$$\Rightarrow g' = -\alpha g$$
$$\Rightarrow \cancel{A}\lambda\cos\lambda a = -\alpha\cancel{A}\sin\lambda a$$
$$\Rightarrow -\cot\lambda a = \frac{\alpha}{\lambda}$$

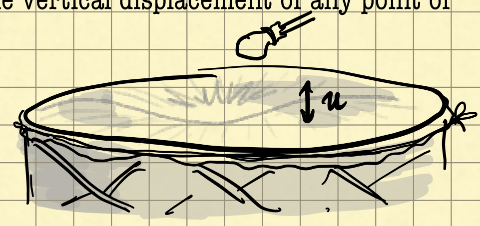
This is a trascendental equation for λ.
Solving it is equivalent to finding the intersections between $-\cot\lambda a$ and α/λ. For example, for $a=1$, $\alpha=0.5$:



$$\therefore\ T(t,z) = T_0 e^{-\lambda^2\beta t}\sin(\lambda_i(a,\alpha)\, z)$$

determined by initial conditions (not given in the problem).

Problem: consider a circular membrane (like the one in a drum) of radius $r=1$. The vertical displacement of any point of the membrane $u(t,r,\varphi)$ follows the wave differential equation



$$\nabla^2 u(t,r,\varphi) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}u(t,r,\varphi)$$

Solve for u with the following boundary conditions:

- The membrane is held in its place at the external boundary, i.e., $u(t, r=1, \varphi) = 0$
- The initial solution is the least oscillatory possible

First, we take the wave equation and separate the time variable from the spatial part

Ansatz: $u = F(r,\varphi)\,T(t)$

$$\Rightarrow T\nabla^2 F = \frac{1}{v^2} F T'' \quad \Big| \cdot \frac{1}{FT}$$

$$\Rightarrow \frac{\nabla^2 F}{F} = \frac{1}{v^2}\frac{T''}{T} = -K^2$$: const. to be det.

- Time:

$$T'' + K^2 c^2 T = 0$$

$$\Rightarrow T(t) \in \{\cos(K v t + \alpha)\}$$

- Space:

$$\nabla^2 F + K^2 F = 0$$

the separation of variables (time-space) has given us the Helmholtz differential equation. Now we do another separation of variables by writing the differential operator in cylindrical coordinates (=polar in this case because it's only in r and φ)

$$F(r,\varphi) = R(r)\,\Phi(\varphi)$$

$$\xrightarrow{\dots} \frac{1}{rR}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{1}{r^2\Phi}\frac{\partial^2\Phi}{\partial\varphi^2} + K^2 = 0 \quad \Big| \cdot r^2$$

$$\frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + K^2 r^2 = \frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\varphi^2} = -n^2$$: const to be det.

$\underline{\underline{\Phi}}$:

$$\frac{\partial^2\Phi}{\partial\varphi^2} = -n^2\Phi$$

$$\Phi \in \{\sin(n\varphi),\ \cos(n\varphi)\}$$

$\underline{\underline{R}}$:

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + (K^2 r^2 - n^2)R = 0$$

Bessel ODE

$$\Rightarrow R(r) \in \{J_n(kr),\ \cancel{Y_n(kr)}\}$$ $\rightarrow 0$ singular at $r=0$.

- Least oscillatory $\Rightarrow$ $n = 0$.

$\longrightarrow \Phi \in \{1\}$ $\qquad\qquad\qquad$ $\longrightarrow R(r) \in \{J_0(kr)\}$

- Boundary condition: at $r = 1$, $u = 0 \Rightarrow R(1) = 0$.

$$\Rightarrow J_0(k) = 0$$

$$\Rightarrow k = C_{0m} \quad , \quad m \in \mathbb{N}.$$

oscillation modes.
The least oscillatory mode is $m = 1$

$$\therefore \quad u(t, r, \varphi) = \overset{\text{const.}}{A}\, T(t) R(r) \Phi(\varphi)$$

$$= A \cos(C_{01} v t + \alpha) J_0(C_{01} r)$$

As an additional initial condition, imagine we say at t = 0, the drum was hit by a stick (see drawing at the beginning of the exercise) and the membrane was deformed downwards following the Bessel function and the maximum displacement was 0.3. Then, the solution for the least oscillatory mode is

at $t=0$, $\quad u(t=0, r, \varphi) = A \cos(\alpha) \underbrace{J_0(C_{01} \cdot 0)}_{=1}$

we choose $\alpha = 0$
$A = -0.3$

$$\therefore u(t, r, \varphi) = -0.3 \cos(C_{01} v t) J_0(C_{01} r)$$

with C_{01} = scipy.special.jn_zeros(0,1) ≈ 2.405