Problem: compute the Fourier transform of the following functions using the definition:

1) The Dirac delta $\delta(t-t_0)$

$$\mathcal{F}\{\delta(t-t_0)\}(\omega) = \int_{-\infty}^{\infty} \delta(t-t_0)e^{-i\omega t}\,dt = e^{-i\omega t_0} = F(\omega)$$

2) The step function with a decreasing exponential

$$f:\mathbb{R}\to\mathbb{R}, \quad f(t) = \begin{cases} e^{-at}, & t \geqslant 0 \\ 0, & t<0 \end{cases} = e^{-at}H(t) \qquad \text{with } a \geqslant 0$$

$$\mathcal{F}\{e^{-at}H(t)\} = \int_0^{\infty} e^{-(a+i\omega)t}\,dt = \lim_{b\to\infty}\int_0^b e^{-(a+i\omega)t}\,dt$$

let $u = -(a+i\omega)t \Rightarrow du = -(a+i\omega)\,dt$

$$\Rightarrow \lim_{b\to\infty} \frac{-1}{a+i\omega}\left[e^{-(a+i\omega)b} - e^{0}\right] = \frac{1}{a+i\omega} = F(\omega)$$

$$\underbrace{\underbrace{e^{-a\infty}}_{\text{real decreasing envelope}} \cdot e^{-i\omega\infty}}_{\to 0}$$

limit doesn't exist (oscillatory behavior)

3) The Heaviside or step function

$$H(t):\mathbb{R}\to\mathbb{R}, \quad H(t) = \begin{cases} 1, & t>0 \\ 0, & t<0 \end{cases}$$

$$\mathcal{F}\{H(t)\}(\omega) = \int_0^{\infty} e^{-i\omega t}\,dt = \lim_{b\to\infty}\frac{-1}{i\omega}\left[e^{-i\omega t}\right]_0^b$$

the limit $e^{-i\omega\infty}$ doesn't exist (oscillations at infinity). But we can use the result from the previous example, thinking of the Heaviside function as a "limiting function" of the decreasing exponential step function when $a \to 0$

Then,

$$\mathcal{F}\{H(t)\}(\omega) = \frac{1}{i\omega}, \quad \text{iff } \omega \neq 0.$$

But if we compute the inverse transform by the definition,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{i\omega}e^{i\omega t}\,d\omega$$

we see that we must include the case $\omega = 0$.

$$\frac{1}{2\pi}\lim_{b\to 0^+}\int_b^{\infty}\frac{1}{i\omega}e^{i\omega t}\,d\omega + \frac{1}{2\pi}\lim_{b\to 0^-}\int_{-\infty}^{b}\frac{1}{i\omega}e^{i\omega t}\,d\omega + \frac{1}{2\pi}\int_{0^-}^{0^+} f_1(\omega)e^{i\omega t}\,d\omega$$

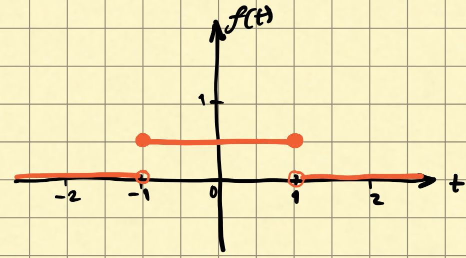
so we need a function f_1 that only exists at $\omega=0$, such that the last integral is finite. This is the Dirac delta. Then we have

$$\mathcal{F}\{H(t)\}(\omega) = \text{P.V.}\frac{1}{i\omega} + \pi\delta(\omega)$$

where p.v. means Cauchy's principal value.

4) The rectangular function

$$f:\mathbb{R}\to\mathbb{R}, \quad f(t)=\begin{cases} 1/2, & |t|\leq 1 \\ 0, & |t|>1 \end{cases}$$



$$\mathcal{F}\{f(t)\}(\omega)=\frac{1}{2}\int_{-1}^{1} e^{-i\omega t}\,dt$$

- $\omega=0$:

$$=\frac{1}{2}\int_{-1}^{1} dt = 1$$

- $\omega\neq 0$:

$$=\frac{1}{2}\int_{-1}^{1} e^{-i\omega t}\,dt=\frac{1}{2}\left[\frac{-1}{i\omega}e^{-i\omega t}\right]_{-1}^{1}=\frac{1}{\omega}\left[\frac{e^{i\omega}-e^{-i\omega}}{i}\right]=\frac{\sin\omega}{\omega}$$

$$\therefore \mathcal{F}\{f(t)\}(\omega)=\begin{cases} 1, & \omega=0 \\ \frac{\sin\omega}{\omega}, & \omega\neq 0 \end{cases} := \operatorname{sinc}\omega.$$

Problem: show the property of similarity of the Fourier transform.

$$\mathcal{F}\{f(ct)\}=\frac{1}{|c|}F\left(\frac{\omega}{c}\right)$$

$$\mathcal{F}\{f(ct)\}(\omega)=\int_{-\infty}^{\infty} f(ct)\,e^{-i\omega t}\,dt=\left(\begin{array}{l}\text{let } u=ct\\ \Rightarrow du=c\,dt\end{array}\right)=\int_{-\infty}^{\infty} f(u)\,e^{-i\omega u/c}\,\frac{du}{|c|}=\frac{1}{|c|}F\left(\frac{\omega}{c}\right)$$

limits don't change order only if there is an absolute value.

Problem: show the property of displacement in time of the Fourier transform.

$$\mathcal{F}\{f(t-a)\}=e^{-i\omega a}F(\omega)$$

$$\int_{-\infty}^{\infty} f(t-a)\,e^{-i\omega t}\,dt=\left(\begin{array}{l}\text{let } u=t-a\\ t=u+a\\ du=dt\end{array}\right)=\int_{-\infty}^{\infty} f(u)\,e^{-i\omega(u+a)}\,du=\int_{-\infty}^{\infty} f(u)\,e^{-i\omega a}\cdot e^{-i\omega u}\,du$$

$$=e^{-i\omega a}F(\omega)$$

Problem: show the property of the derivative in frequencies of the Fourier transform.

$$\mathcal{F}\{t f(t)\}=iF'(\omega)$$

$$F(\omega)=\int_{-\infty}^{\infty} f(t)\,e^{-i\omega t}\,dt \Rightarrow \frac{d}{d\omega}F(\omega)=\frac{d}{d\omega}\int_{-\infty}^{\infty} f(t)\,e^{-i\omega t}\,dt \Rightarrow F'(\omega)=\int_{-\infty}^{\infty} -i\,t f(t)\,e^{-i\omega t}\,dt$$

$$\Rightarrow iF'(\omega)=\int_{-\infty}^{\infty} t f(t)\,e^{-i\omega t}\,dt \Rightarrow \mathcal{F}\{t f(t)\}=iF'(\omega).$$

Problem: Compute the Fourier transform of the function $e^{-at}\sin\omega_0 t H(t)$, $H(t)$: Heaviside

$$\mathcal{F}\left\{e^{-at}\sin\omega_0 t\, H(t)\right\}$$

$$\mathcal{F}\left\{e^{-at}\cdot\frac{1}{2i}\cdot\left(e^{i\omega_0 t}-e^{-i\omega_0 t}\right)H(t)\right\}$$

$$=\frac{1}{2i}\mathcal{F}\left\{e^{-at}e^{i\omega_0 t}H(t)\right\}-\frac{1}{2i}\mathcal{F}\left\{e^{-at}e^{-i\omega_0 t}H(t)\right\}$$

De la tabla, sabemos que $\mathcal{F}\{e^{-at}H(t)\}=\frac{1}{a+i\omega}$. También, que

$\mathcal{F}\{e^{i\omega_0 t}f(t)\}=\mathcal{F}\{f(t)\}(\omega-\omega_0)=F(\omega-\omega_0)$

$$\frac{1}{i}=\frac{1}{\sqrt{-1}}=\frac{\sqrt{-1}}{\sqrt{-1}}\cdot\frac{1}{\sqrt{-1}}=-\sqrt{-1}=-i$$

$$\Rightarrow\quad\frac{1}{2i}\,\frac{1}{a+i(\omega-\omega_0)}-\frac{1}{2i}\,\frac{1}{a+i(\omega+\omega_0)}$$

$$=-\frac{i}{2}\left[\frac{\cancel{a}+i(\omega+\omega_0)-\cancel{a}-i(\omega-\omega_0)}{[a+i(\omega-\omega_0)][a+i(\omega+\omega_0)]}\right]$$

$$=\frac{-i}{2}\,\frac{i\cancel{\omega}+i\omega_0-i\cancel{\omega}+i\omega_0}{a^2+ai(\omega+\cancel{\omega_0})+ai(\omega-\cancel{\omega_0})-(\omega^2-\omega_0^2)}$$

$$=\frac{-i}{\cancel{2}}\,\frac{\cancel{2}i\omega_0}{a^2+2ai\omega-\omega^2+\omega_0^2}$$

$$=\frac{\omega_0}{(a+i\omega)^2+\omega_0^2}$$

Problem: solve the differential equation

$$\ddot{x}+\omega_0^2x=0\quad,\quad\omega_0>0$$

which describes a harmonic oscillator using Fourier transforms.

$$\mathcal{F}\{\ddot{x}\}+\mathcal{F}\{\omega_0^2x\}=0$$

$$i^2\omega^2X(\omega)+\omega_0^2X(\omega)=0$$

$$\Rightarrow\quad(\omega_0^2-\omega^2)X(\omega)=0$$

$$\Rightarrow\quad(\omega^2-\omega_0^2)X(\omega)=0$$

$$\Rightarrow\quad(\omega+\omega_0)(\omega-\omega_0)X(\omega)=0$$

Por las propiedades de la delta,

si $X(\omega)=\delta(\omega-\omega_0)\cdot 2\pi$

$$\Rightarrow\quad 2\pi(\omega-\omega_0)\delta(\omega-\omega_0)=0$$

entonces,

$$X(\omega)=2\pi\delta(\omega-\omega_0)\cdot A$$

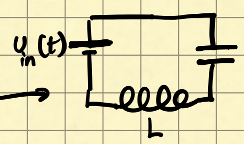
$$\Rightarrow\quad x(t)=\mathcal{F}^{-1}\{A2\pi\delta(\omega-\omega_0)\}$$

$$x(t)=Ae^{-i\omega_0 t}$$

Nota: una suma de deltas también es solución.

Problem: solve the differential equation

$$\ddot{x} + \omega_0^2 x = \cos \omega_1 t$$



voltage

$$\ddot{v}(t) + \underbrace{\frac{1}{LC}}_{\omega_0^2} v(t) = \overbrace{\cos(\omega_1 t)}^{v_{in}(t)/LC}$$

which represents a forced harmonic oscillator or a LC circuit, using Fourier transforms.

$$\mathcal{F}\{\ddot{x}\} + \mathcal{F}\{\omega_0^2 x\} = \mathcal{F}\{\cos \omega_1 t\}$$

$$i^2\omega^2 X(\omega) + \omega_0^2 X(\omega) = \pi[\delta(\omega-\omega_1) + \delta(\omega+\omega_1)]$$

$$-\omega^2 X(\omega) + \omega_0^2 X(\omega) = \pi[\delta(\omega-\omega_1) + \delta(\omega+\omega_1)]$$

$$X(\omega)(\omega_0^2 - \omega^2) = \pi[\delta(\omega-\omega_1) + \delta(\omega+\omega_1)]$$

$$X(\omega) = \frac{\pi[\delta(\omega-\omega_1) + \delta(\omega+\omega_1)]}{\omega_0^2 - \omega^2}$$

$$F(\omega) = \frac{1}{\omega_0^2 - \omega^2} \qquad G(\omega) = [\delta(\omega-\omega_1) + \delta(\omega+\omega_1)]\cdot\pi$$

Convolución

$$x(t) = \mathcal{F}^{-1}[F(\omega)G(\omega)] = \int_{-\infty}^{\infty} f(t-\tau) g(\tau)\, dt$$

en este caso, intentamos directamente

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} F(\omega) G(\omega)\, d\omega$$

$$= \frac{\cancel{\pi}}{2\cancel{\pi}}\int_{-\infty}^{\infty} e^{i\omega t} \cdot \frac{1}{\omega_0^2 - \omega^2} \cdot [\delta(\omega-\omega_1) + \delta(\omega+\omega_1)]\, d\omega$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{i\omega t} \frac{1}{\omega_0^2 - \omega^2} \delta(\omega-\omega_1)\, d\omega + \frac{1}{2}\int_{-\infty}^{\infty} e^{i\omega t} \frac{1}{\omega_0^2 - \omega^2} \delta(\omega+\omega_1)\, d\omega$$

$$= \frac{1}{2}\left[e^{i\omega_1 t} \frac{1}{\omega_0^2 - \omega_1^2} + e^{-i\omega_1 t} \frac{1}{\omega_0^2 - \omega_1^2} \right] = \frac{\cos(\omega_1 t)}{\omega_0^2 - \omega_1^2}$$