$\cos x \quad T = 2\pi$

$S \quad \cos\left(\frac{2\pi}{S}x\right)$

si $S=\pi$ $\quad \cos(2x)$

Problem. a) What is the period of the following functions?

$\sin(x) : 2\pi \qquad \cos(2x): \pi \qquad \sin(2\pi x): 1$

b) Change the period of the following functions from 2π to T'.

$\sin(x) : \quad T = 2\pi \quad \sin\left(\frac{2\pi x}{T'}\right)$

$\cos(x) : \quad T = 2\pi \quad \cos\left(\frac{2\pi}{T'}x\right)$

$e^{ix} : \quad T = 2\pi \quad e^{i\frac{2\pi}{T'}x}$

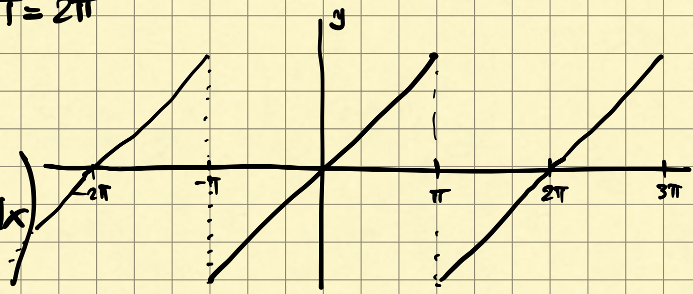
$f(x)$, period T

$g(x)$, period S

$$\underline{g(x+S)} = f\left(\frac{T}{S}(x+S)\right) = f\left(\frac{T}{S}x + T\right) = f\left(\frac{T}{S}x\right) = \underline{g(x)} \checkmark$$

Problem. Compute the real Fourier series of the periodic function $f: [-\pi, \pi[\longrightarrow \mathbb{R}$, $f(x) = x$.

$T = 2\pi$



$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = \frac{1}{2\pi}x^2\Big|_{-\pi}^{\pi} = 0$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(kx)\,dx = \frac{1}{\pi}\left(\frac{x}{k}\sin(kx)\Big|_{-\pi}^{\pi} - \frac{1}{k}\int_{-\pi}^{\pi}\sin(kx)\,dx\right)$$

$$= \frac{1}{\pi}\left(\frac{1}{k^2}\cos(kx)\Big|_{-\pi}^{\pi}\right) = \frac{1}{k^2\pi}\left((-1)^k - (-1)^k\right) = 0$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(kx)\,dx = \frac{1}{\pi}\left(-\frac{x}{\pi}\cos(kx)\Big|_{-\pi}^{\pi} + \frac{1}{k}\int_{-\pi}^{\pi}\cos(kx)\,dx\right)$$

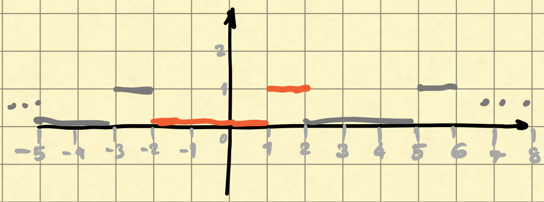
$$= \frac{1}{\pi}\left(-\frac{\pi}{k}(-1)^k - \frac{\pi}{k}(-1)^k\right) = \frac{2}{k}(-1)^{k+1}$$

$$x = \sum_{k=1}^{\infty}\frac{2}{k}(-1)^{k+1}\sin(kx) = 2\left(\sin(x) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} - \cdots\right)$$

en el intervalo $[-\pi, \pi]$

Problem: Compute the first terms of the complex Fourier series of the periodic function $f : [-2, 2[\to \mathbb{R},\ f(x) = H(x-1)$

A plot of the function:



Fourier expansion:

$$F(x) = \sum_{k=-\infty}^{\infty} c_k \, e^{ik\frac{2\pi}{T}x}$$

where

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} f(x)\, e^{-ik\frac{2\pi}{T}x}\, dx \quad , \quad k \in \mathbb{Z}.$$

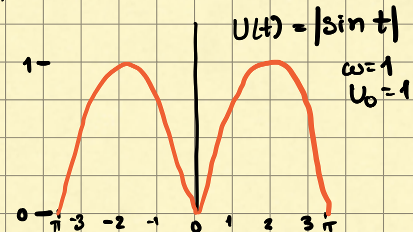
Here, $T = 4$.

$$c_k = \frac{1}{4}\int_{-2}^{2} H(x-1)\, e^{-ik\pi x/2}\, dx = \frac{1}{4}\int_{1}^{2} e^{-ik\pi x/2}\, dx = \begin{bmatrix} \text{let } u = -ikx/2 \\ du = -i\frac{k}{2}dx \end{bmatrix} = \frac{-1}{4}\cdot\frac{2}{ik}\int_{\dots}^{\dots} e^{u}\, du$$

$$= \frac{-1}{2ik}\, e^{u}\Big|_{\dots}^{\dots} = \frac{-1}{2ik}\, e^{-ikx/2}\Big|_{1}^{2} = \frac{-1}{2ik}\left[e^{-ik} - e^{-ik/2}\right]$$

$$\Rightarrow \quad H(x-1) = \sum_{k=-\infty}^{\infty} \frac{-1}{2ik}\left[e^{-ik} - e^{-ik/2}\right] e^{ik\pi x/2} \quad , \text{ in the interval } [-2, 2[.$$

Problem: In a circuit, AC power can be transformed in DC power starting from a periodic signal, like $U_c(t) \propto \sin(\omega t)$ and using electronic components to invert the negative part, so we get $U_{in}(t) = U_0\,|\sin(\omega t)|$
Below is a plot of this function.



The real Fourier series expansion of this function is found to be

$$|\sin(\omega t)| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2k\omega t)}{4k^2-1}$$

Express this result in terms of a complex Fourier series.

We know that

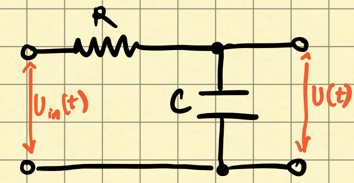
$$c_k = \frac{a_k - i\cancel{b_k}^{\,0}}{2} \qquad c_{-k} = \frac{a_k + i\cancel{b_k}^{\,0}}{2} \qquad c_0 = \frac{a_0}{2}$$

$$\frac{a_0}{2} = \frac{2}{\pi} \qquad a_k = \frac{1}{4k^2-1}\cdot\frac{4}{\pi} \qquad b_k = 0$$

$$c_k = \frac{1}{4k^2-1}\frac{2}{\pi} \qquad c_{-k} = \frac{1}{4k^2-1}\frac{2}{\pi} \qquad c_0 = \frac{2}{\pi}$$

$$\Rightarrow \quad |\sin(\omega t)| = \sum_{k=-\infty}^{\infty} \frac{1}{4k^2-1}\frac{2}{\pi}\, e^{2ik\omega t}$$

Problem: Consider the circuit in the figure. It is a low-pass filter made from a resistor and a capacitor. The differential equation that describes the out voltage $U: \mathbb{R} \to \mathbb{R},\ U(t)$



$$RC \frac{d}{dt} U(t) + U(t) = U_{in}(t)$$

where $R, C \in \mathbb{R}, R, C > 0$ and the function $U_{in}: \mathbb{R} \to \mathbb{R},\ U_{in}(t)$ is the in voltage.

Consider the case where $U_{in}(t) = U_0 |\sin(\omega t)|$, $U_0 > 0$ Solve the differential equation using Fourier series.

Ansatz $U(t) = \sum_{k=-\infty}^{\infty} d_k e^{2i\omega kt}$, we know that $|\sin(\omega t)| = \sum_{k=-\infty}^{\infty} \frac{1}{4k^2-1} \frac{2}{\pi} e^{2ik\omega t}$

$$RC \frac{d}{dt} U(t) + U(t) = U_{in}(t)$$

$$\Rightarrow U(t) = \sum_{k=-\infty}^{\infty} d_k e^{2i\omega t k}$$

$$\frac{dU(t)}{dt} = \sum_{k=-\infty}^{\infty} d_k \cdot 2i\omega k\, e^{2i\omega kt}$$

$$\sum_{k=-\infty}^{\infty} 2RC\, i\omega k\, d_k e^{2i\omega kt} + \sum_{k=-\infty}^{\infty} d_k e^{2i\omega kt} = \frac{2U_0}{\pi} \sum_{k=-\infty}^{\infty} \frac{1}{4k^2-1} e^{2i\omega kt}$$

Comparing coefficients,

$$2RC\, i\omega k\, d_k + d_k = \frac{2U_0}{\pi} \frac{1}{4k^2-1}$$

$$d_k (2RC i\omega k + 1) = \frac{2U_0}{\pi} \frac{1}{4k^2-1}$$

(check: indep. of t)

$$d_k = \frac{1}{2RC i\omega k + 1} \frac{2U_0}{\pi} \frac{1}{4k^2-1}$$

$$\Rightarrow U(t) = \sum_{k=-\infty}^{\infty} \frac{1}{2RC i\omega k + 1} \frac{2U_0}{\pi} \frac{1}{4k^2-1} e^{2i\omega kt}$$

Problem: prove the formulae for changing the coefficients from a real Fourier series to a complex Fourier series.

$$f(x) = f(x)$$

$$\sum_{k=-\infty}^{\infty} c_k e^{ikx} = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

$$\sum_{k=-\infty}^{-1} c_k e^{ikx} + c_0 + \sum_{k=1}^{\infty} c_k e^{ikx} = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

let $k \to -k$

$$\sum_{k=1}^{\infty} c_{-k} e^{-ikx} + c_0 + \sum_{k=1}^{\infty} c_k e^{ikx} = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

$$c_0 + \sum_{k=1}^{\infty} \left(c_{-k} e^{-ikx} + c_k e^{ikx}\right) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

$$c_0 = \frac{a}{2}$$

$$c_{-k} e^{-ikx} + c_k e^{ikx} = a_k \left(\frac{e^{ikx} + e^{-ikx}}{2}\right) + b_k \left(\frac{e^{ikx} - e^{-ikx}}{2i}\right)$$

$$c_{-k} e^{-ikx} + c_k e^{ikx} = \frac{a_k e^{ikx}}{2} + \frac{a_k e^{-ikx}}{2} + \frac{b_k e^{ikx}}{2i} - \frac{b_k e^{-ikx}}{2i}$$

$$c_k = \frac{a_k}{2} + \frac{b_k}{2i} = \frac{a_k - ib_k}{2} \qquad c_0 = \frac{a_0}{2}$$

$$c_{-k} = \frac{a_k}{2} - \frac{b_k}{2i} = \frac{a_k + ib_k}{2}$$