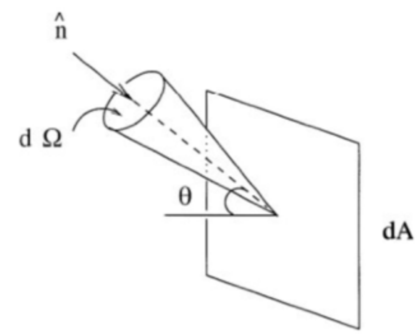
Notes on Radiative transfer

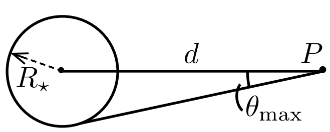
Radiative transfer

Blackbody radiation: Planck's law: the energy density U_ν for the frequency ν (formally, the frequency range, $\nu, \nu + d\nu$) is $U_\nu d\nu = \dfrac{8\pi h}{c^3} \dfrac{\nu^3 d\nu}{\exp\left[h\nu/(k_B T)\right] - 1} := \dfrac{4\pi}{c} B_\nu(T)$



Specific intensity. The energy of a given frequency coming from a solid angle $d\Omega$ and inciding over an area dA (projected perpendicularly to it) in a time dt is $dE_\nu d\nu = I_\nu(\vec{\mathbf{r}}, t, \hat{\mathbf{n}}) \cos\theta \, dA \, dt \, d\Omega \, d\nu$, where I_ν is the specific intensity.

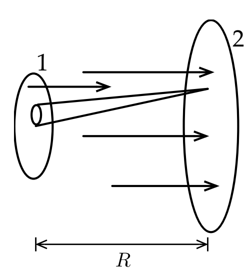
Radiation flux for a frequency ν: $F_\nu = \displaystyle\int I_\nu \cos\theta d\Omega$ (power that crosses an area from all directions). The total flux for all frequencies is $F = \displaystyle\int F_\nu d\nu$.



Uniformly radiating sphere: consider a sphere of radius $R_\star$ that radiates with uniform intensity I_0. z-axis along d. The flux that an observer located at P receives is $F = \displaystyle\int I \cos\theta d\Omega = I_0 \int_0^{2\pi} d\phi \int_0^{\theta_{\max}} \cos\theta \sin\theta d\theta = I_0 \cdot \frac{1}{2} \sin^2\theta_{\max} \cdot 2\pi$, but since $\sin\theta_{\max} = R_\star/d$, $\implies F = I_0 \pi (R_\star/d)^2$. Observe that the term $\pi(R_\star/d)^2$ is the solid angle subtended by the sphere at P (both making the star bigger or closer have the same effect).

Energy density: energy dE_ν that passes through a cylinder of base dA and height cdt: $\dfrac{dE_\nu}{\cos\theta dA \, cdt} = \dfrac{I_\nu}{c} d\Omega \implies U_\nu = \displaystyle\int \frac{I_\nu}{c} d\Omega$; if isotropic, $U_\nu = \dfrac{4\pi}{c} I_\nu$

Radiation pressure. Radiation has momentum dE_ν/c.
pressure $= \dfrac{\text{force}_\perp}{dA} = \dfrac{\text{momentum}}{dtdA} = \dfrac{dE_\nu \cos\theta}{c} \dfrac{1}{dAdt} = \dfrac{I_\nu}{c} \cos^2\theta d\Omega$. The pressure counting all directions is $P_\nu = \dfrac{1}{c} \displaystyle\int I_\nu \cos^2\theta d\Omega$; if isotropic, $P_\nu = \dfrac{4\pi}{3} \dfrac{I_\nu}{c}$. Comparing with the isotropic energy density, $P_\nu = \dfrac{1}{3} U_\nu$.



Radiative transfer in empty space: the radiation going through 2 due to 1 is $I_{\nu 2} dA_2 dt d\Omega_2 d\nu$, and, by symmetry, the radiation that 1 receives due to 2 is $I_{\nu 1} dA_1 dt d\Omega_1 d\nu$. But the energy is conserved, and $d\Omega = dA/R^2$, so, $I_{\nu 1} = I_{\nu 2}$, which implies that, along the ray path, $\dfrac{dI_\nu}{ds} = 0$ in empty space. *Conclusion*: the specific intensity does not depend on distance, so it's a measure of surface brightness (total $I \propto r^{-2}$ but the angular size also is $\propto r^{-2}$, so the effects cancel out).

Radiative transfer equation: in the presence of matter, $\frac{dI_\nu}{ds} = j_\nu - \alpha_\nu I_\nu$. If matter emits, it sums j_ν (emission coefficient, units of intensity/length). If matter absorbs, it will diminish the intensity, so one subtracts an amount proportional to I_ν; α_ν is the absorption coefficient.

Optical depth: if we consider absorption only, $dI_\nu/ds = -\alpha_\nu I_\nu \implies dI_\nu/I_\nu = \alpha_\nu ds$. Integrating in a path, we get $\ln\left[\frac{I_\nu(s)}{I_\nu(s_0)}\right] = -\int_{s_0}^{s} \alpha_\nu(s')ds'$. We define the integral in the rhs as the optical depth τ_ν. Then, the solution for only absorption is $I_\nu(s) = I_\nu(s_0)e^{-\tau_\nu}$. If the optical depth is $\ll 1$, the intensity at the beginning is almost the same as at the end $\Longrightarrow$ the medium is optically thin or transparent. If the optical depth is $\gg 1$, the final intensity is about zero $\Longrightarrow$ the medium is optically thick or opaque.

Source function: we define $S_\nu := j_\nu/\alpha_\nu$. We can have the radiative transfer equation in terms of S_ν, τ_ν: $\frac{1}{\alpha_\nu}\frac{dI_\nu}{ds} = \frac{j_\nu}{\alpha_\nu} - I_\nu \Longrightarrow \frac{dI_\nu}{d\tau_\nu} = S_\nu + I_\nu$. This differential equation can be solved by using an integrating factor e^{τ_ν}: $\frac{d}{dt}(I_\nu e^{\tau_\nu}) = S_\nu e_\nu^{\tau}$; we integrate from $s_0 < s' < s$ in the ray path, which corresponds to $0 < \tau'_\nu < \tau_\nu$, yielding $I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau'_\nu)} S_\nu(\tau'_\nu)d\tau'_\nu$. $I_\nu(0)$ is the boundary condition and can be fixed by including incoming radiation. $I_\nu(\tau_\nu)$ is the intensity emerging from the medium to the observer; the first term is absorption along the ray, and the second term is the sum of the contributions of all sources along the ray.

Limiting cases: for constant coefficients and no incoming radiation, $I_\nu(\tau_\nu) = S_\nu(1 - e^{-\tau_\nu})$. If the object is optically thin, $\tau_\nu \ll 1$, $e^{-\tau_\nu} \approx 1 - \tau_\nu$, with $\tau_\nu = \alpha_\nu L$, being L the length of the material $\Longrightarrow I_\nu = j_\nu L$. If the object is optically thick, $\tau_\nu \gg 1 \implies I_\nu = S_\nu$.

Kirchhoff's law: when matter is in thermodynamic equilibrium, all that is emitted must be absorbed. That means that $j_\nu = \alpha_\nu I_\nu$, but since $I_\nu = B_\nu$ for equilibrium, $S_\nu = j_\nu/\alpha_\nu = \alpha_\nu I_\nu/\alpha_\nu = B_\nu(T)$.