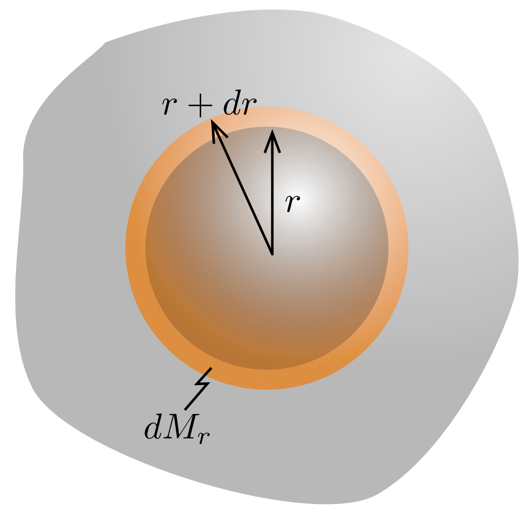
Physics of massive stars

Contents

Stellar structure equations

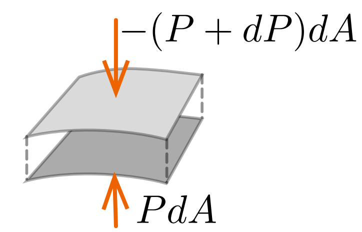
I. Conservation of mass:



The matter enclosed in a spherical shell of thickness dr has a mass $dM_r = 4\pi r^2 \rho dr$. The first stellar structure equation is $\frac{dM_r}{dr} = 4\pi r^2 \rho$.

II. Hydrostatic equilibrium: net force caused by the gradient of pressure: $-dPdA$. $\sum F_r = 0 \implies -dPdA - \frac{GM_r}{r^2}\rho dr dA = 0$.

The second stellar structure equation is $\frac{dP}{dr} = -\frac{GM_r}{r^2}\rho$.

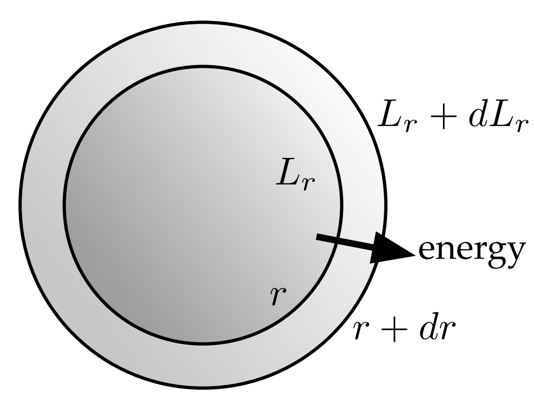


[If the density were constant (obviously not the case), the first structure equation yields $M_r = Mr^3/R^3$, and the second equation, $dP/dr = GMr^3/(r^2R^3)\rho \implies P = GM\rho \int r dr/R^3 = 3GM^2/(8\pi R^4)$]

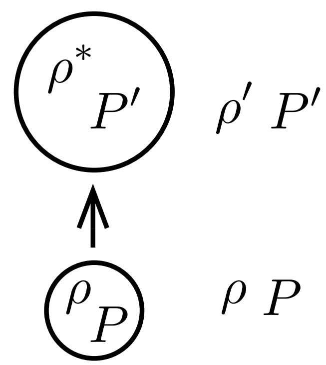
Equation of state: inside of the star, the ideal gas equation of state holds, $PV = nRT$; but $M = \mu n$ (the mass is the mass of one mol times the number of moles) $\implies P = \frac{\rho}{\mu}RT$

III. Energy transport due to nuclear reactions: consider two spherical shells like in the figure, inside a star. Let L_r be the total energy flux per unit time that flows outwards across the shell of radius r. Then, dL_r is the energy input to the energy flux made by the region $[r, r + dr]$. If ε is the rate of energy generation per unit mass per unit time (by nuclear reactions), then,

$dL_r = 4\pi r^2 dr \cdot \rho \cdot \varepsilon \implies \frac{dL_r}{dr} = 4\pi r^2 \rho \varepsilon$ (third structure equation).



IV-a. Energy transport in radiative zones: $U_{\rm rad} = aT^4 \implies \dfrac{dU_{\rm rad}}{dr} = 4aT^3\dfrac{dT}{dr}$. In a radiative zone, the mean free path of radiation is $\lambda = 1/(\kappa\rho)$ [taking Thomson's electron scattering opacity] $\approx 2\,{\rm cm} \ll R_\odot$. That means that radiation behaves like diffusion, and therefore, we can use the diffusion approximation: $F_{\rm rad} = -D\nabla_r U_{\rm rad}$, with the diffusion coefficient $D = c/(3\kappa\rho)$. Therefore, $F_{{\rm rad},r} = -\dfrac{4ac}{3}\dfrac{T^3}{\kappa\rho}\dfrac{dT}{dr}$. Using the luminosity ($L_r = 4\pi r^2 F_{{\rm rad},r}$), we get the energy transfer equation for the radiative regime, which is not valid near the atmosphere (→ § Stellar atmospheres: radiation transport): $\dfrac{dT}{dr} = -\dfrac{3}{16\pi ac}\dfrac{\kappa\rho L}{r^2T^3}$.



Convection inside stars: a blob of material of density ρ, pressure P is in hydrostatic equilibrium with its surroundings, of same density and pressure. The blob is transported adiabatically to another place, where the surrounding density and pressure are ρ', P', respectively. The blob expands/contracts so it has the same pressure than the surroundings, changing its density to ρ^*. If $\rho^* < \rho'$, the blob will be buoyant and continue to move upwards (→ convection), but if $\rho^* < \rho'$, it will return to the original position.

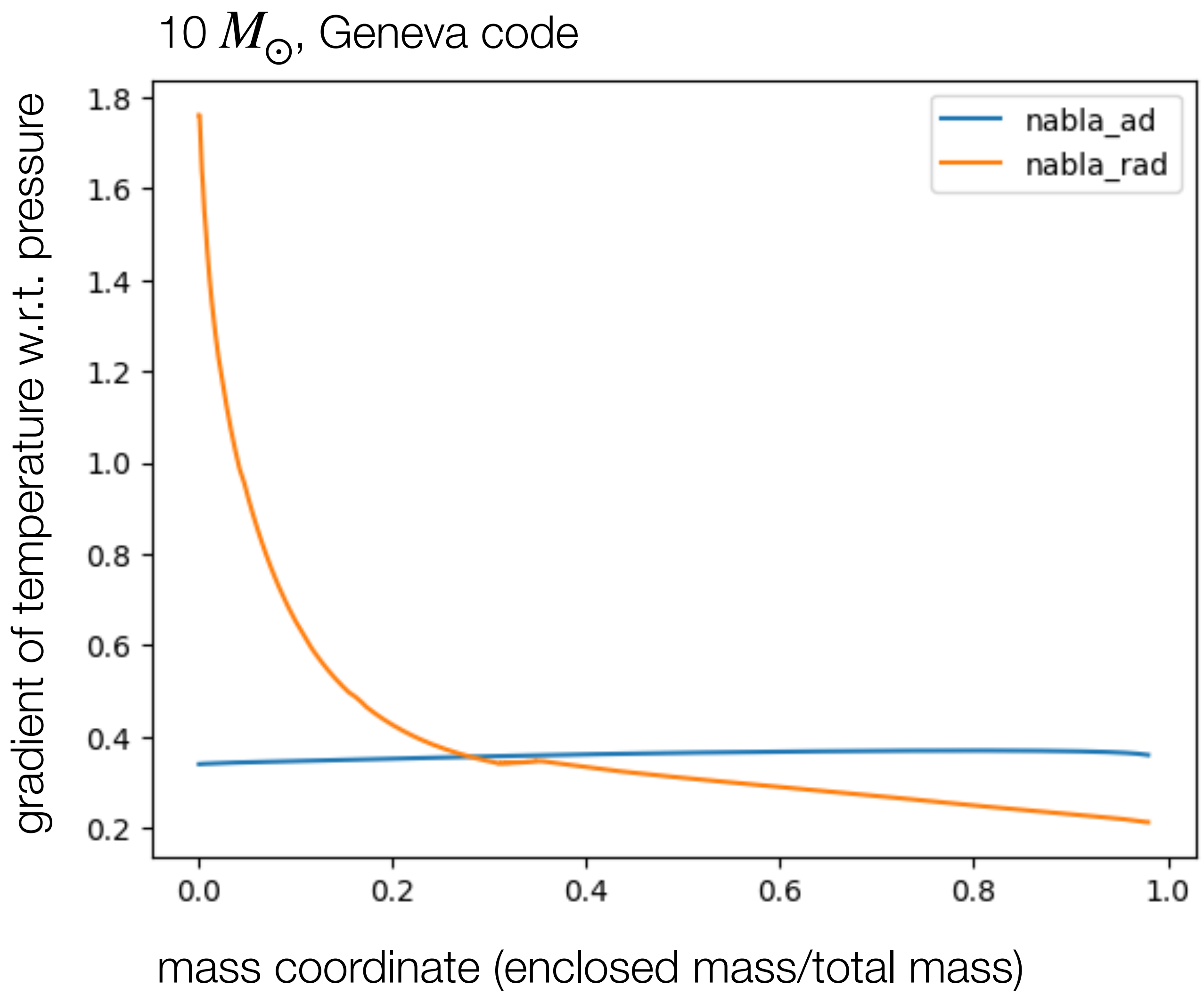
IV-b. Energy transport in convective zones: Since the convection process is adiabatic, $\rho^* = \rho(P'/P)^{1/\gamma}$. Now, $P' = P + \dfrac{dP}{dr}\Delta r$

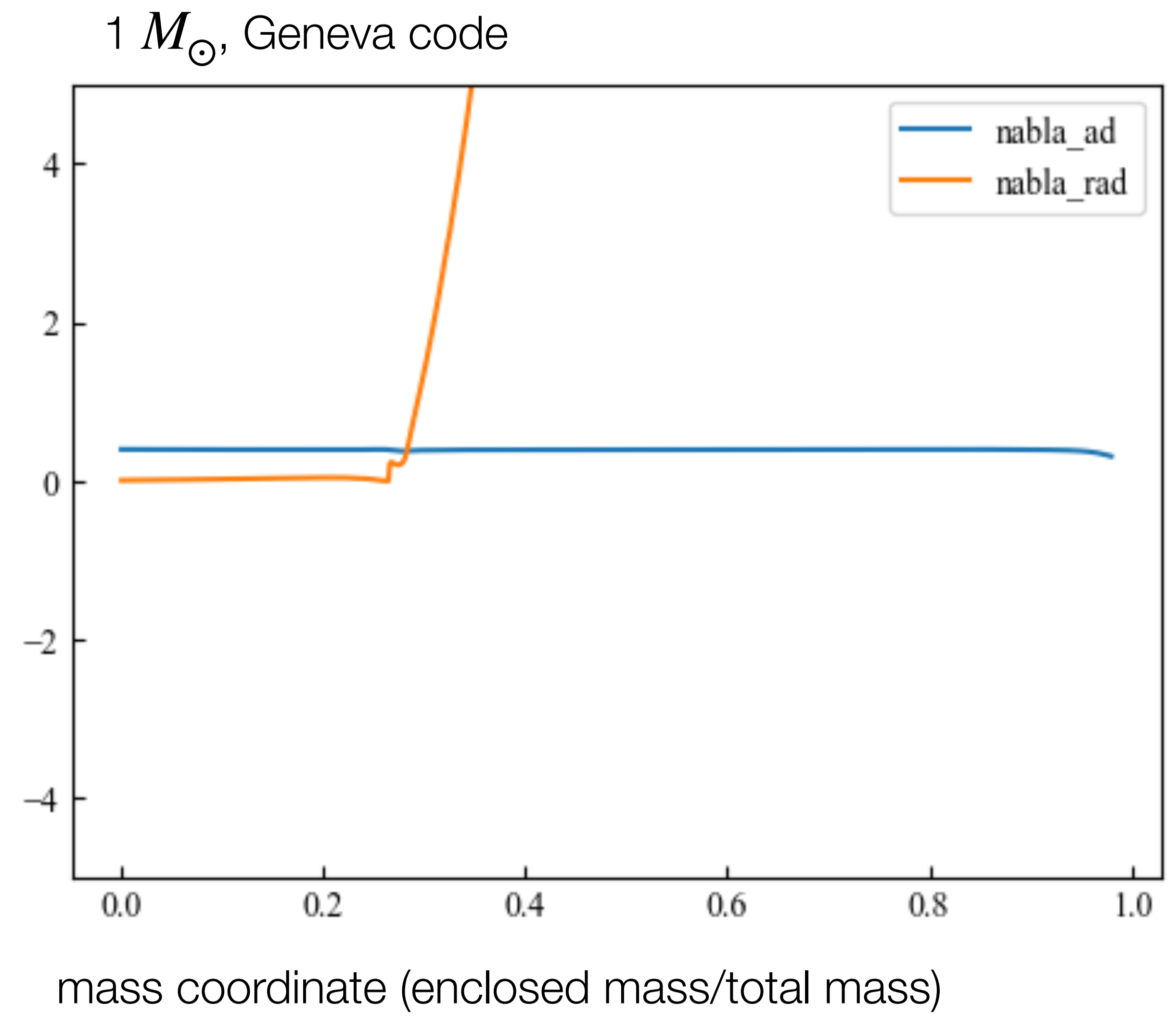
$$\implies \rho^* = \rho\left(1 + \frac{1}{P}\frac{dP}{dr}\Delta r\right)^{1/\gamma}.$$ [Binomial expansion]

$$\implies \rho^* = \rho + \frac{\rho}{\gamma P}\frac{dP}{dr}\Delta r.$$ We can describe the change in the surrounding density field, as well, as $\rho' = \rho + \dfrac{d\rho}{dr}\Delta r$. Using the ideal equation of state $[\rho = P/(RT)]$, we get $\rho' = \rho + \dfrac{1}{RT}\dfrac{dP}{dr}\Delta r - \dfrac{P}{T^2}\dfrac{dT}{dr}\Delta r$

$$\implies \rho' = \rho + \frac{\rho}{P}\frac{dP}{dr}\Delta r - \frac{\rho}{T}\frac{dT}{dr}\Delta r.$$ Finally,

$$\rho^* - \rho' = \left[-\left(1 - \frac{1}{\gamma}\right)\frac{\rho}{P}\frac{dP}{dr} + \frac{\rho}{T}\frac{dT}{dr}\right]\Delta r,$$ which means that the atmosphere is stable if $\left|\dfrac{dT}{dr}\right| < \left(1 - \dfrac{1}{\gamma}\right)\dfrac{T}{P}\left|\dfrac{dP}{dr}\right|$. If the temperature gradient is only slightly steeper than the critical gradient, the typical stellar energy fluxes are obtained. The equation for energy transport in convective zones, is, then, $\dfrac{dT}{dr} = \left(1 - \dfrac{1}{\gamma}\right)\dfrac{T}{P}\dfrac{dP}{dr}$.





Structure of high-mass vs a low-mass star

There is a parameter called $\nabla \equiv \ln T/\ln P$ which describes how temperature changes with pressure (confusing symbol, this is not a vector!). For an idealized radiative and convective regime, one can compute the value of this gradient as $\nabla_{\rm rad}$ and $\nabla_{\rm ad}$, respectively ("ad" means adiabatic because convection is good at trapping heat). The smallest of the two values determines how energy is transported inside the star. In the plots, we have the values of the nablas computed by the Geneva code for a high-mass (left) and a low-mass (right) star. One can easily see that the high-mass star has a convective core ($\nabla_{\rm ad} < \nabla_{\rm rad}$) and radiative envelope ($\nabla_{\rm rad} < \nabla_{\rm ad}$). Exactly the opposite is true in a star like the Sun.

Order-of-magnitude calculations

Mass-density: $\rho \approx \dfrac{3M}{4\pi R^3} \approx 0.24\dfrac{M}{R^3}$

Number density: $n \approx \dfrac{3M}{m_p 4\pi R^3} \approx 0.24\dfrac{M}{m_p R^3}$ (protons dominate mass)

Pressure: from the second structure equation, $P \approx \dfrac{3GM^2}{8\pi R^4} \approx 0.12\dfrac{GM^2}{R^4}$

Main-sequence star: made of ideal gas. Its temperature can be estimated as $T = \dfrac{P}{nk_B} \approx \dfrac{0.12GM^2R^{-4}}{0.24Mm_p^{-1}R^{-3}k_B}$, for the Sun, $T \approx 10^7$ K.

Mass-luminosity relation: order of magnitude version of the mass equation: $\dfrac{M}{R} \sim R^2\rho$ (1), hydrostatic equation: $\dfrac{P}{R} \sim \dfrac{GM\rho}{R^2}$ (2), energy transport equation for radiative zones: $\dfrac{T}{R} \sim \dfrac{K\rho L}{R^2T^3}$ (3) (K is the constant mass-specific absorption coefficient, or opacity), ideal gas $P \sim \rho T/\mu$ (4) (μ is the mean molecular mass, [mass]/mol). Let's substitute the density from (1) everywhere. Then, (3) becomes $T^4 \sim KLM/R^4 \implies L \sim T^4R^4/(KM)$. To substitute T: (4) becomes $P \sim MT/(\mu R^3) \implies T \sim P\mu R^3/M$, but with (2), $T \sim (GM^2/R^4)\,\mu R^3/M \sim G\mu M/R$. Then, we arrive at the mass-luminosity relation, $L \sim G^4\mu^4M^3/K$.

Proportions and stellar lifetime: then, $T \propto M/R$ and $L \propto M^3$. From black body radiation, we know that $L \propto R^2T^4$. Then, comparing luminosities, $M^3 \propto M^2T^2$ or $M \propto T^2$. Using this to substitute for mass in the mass-luminosity relation, $L \propto T^6$. A star is powered by nuclear energy, the amount of which is proportional to the mass. The luminosity of the star is its power output, so $L \propto M/\tau$. Then, the lifetime of a star is $\tau \propto M^{-2}$, which means that more massive stars live shorter lives.

Evolution of massive stars

Nuclear reactions

Proton-proton (PP) chain: (notation: protons nucleons Element) the reaction happens in several steps:

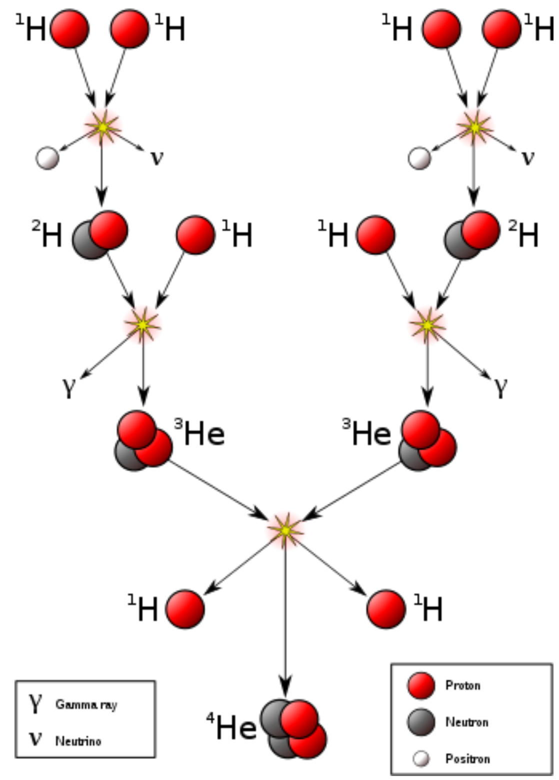
1) $^1_1\text{H} + ^1_1\text{H} \longrightarrow ^2_1\text{H} + e^+ + \nu_e$ (requires β^+ decay: $p + \text{energy} \rightarrow n + e^+ + \nu_e$)

2) $^2_1\text{H} + ^1_1\text{H} \longrightarrow ^3_2\text{He} + \gamma$

3) $^3_2\text{He} + ^3_2\text{He} \longrightarrow ^4_2\text{He} + 2^1_1\text{H}$

overall reaction:

$4^1_1\text{H} \longrightarrow ^4_2\text{He} + 2e^+ + 2\nu_e + 2\gamma + \text{kinetic en.}$



Time that fusion can power a Sun-like star: Since luminosity is power, $t \sim \dfrac{E_{\text{tot}}}{L_\odot}$. In nuclear fusion of hydrogen to helium, $E_{\text{tot}} = N\Delta mc^2$, where $N = M_\odot/(4m_H)$ is the number of 4 hydrogen atoms in the star (4 are needed for the reaction to happen) and $\Delta m \approx 25.71\,\text{MeV}/c^2$ is the difference in mass between 4^1_1H and ^4_2He (it corresponds to the binding energy released in the nuclear reaction). The percentage of mass that transforms into energy is $\varepsilon = \Delta m/(4m_\text{H}) \approx 0.7\,\%$ (sometimes called efficiency). So, $E_{\text{tot}} \sim 0.007 M_\odot c^2 \sim 1.3 \cdot 10^{45}\,\text{J}$ $\implies t \sim \dfrac{1.3 \cdot 10^{45}\,\text{J}}{3.8 \cdot 10^{26}\,\text{J s}^{-1}} \sim 3.3 \cdot 10^{18}\,\text{s} \sim 10^{11}\,\text{yr}$. This assumes a fully convective star (all H available for burning); in reality it is only about 10% for a Sun-like star.

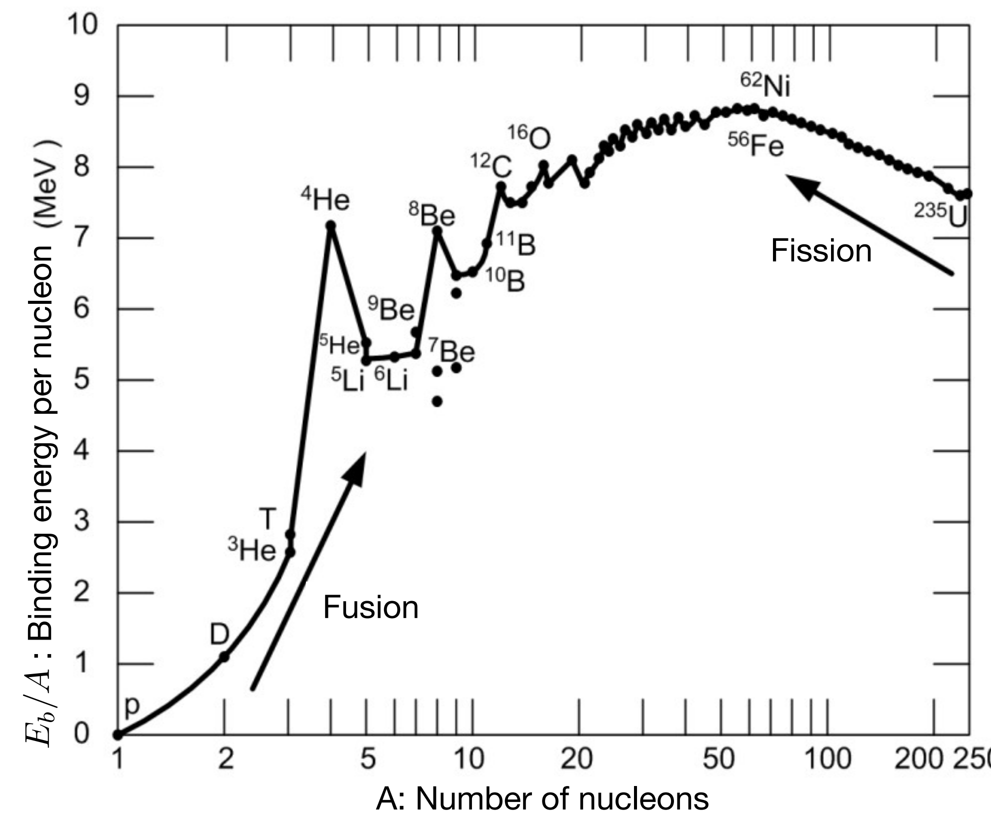
Nuclear potential: Two protons at distances much larger than the nuclear radius repel each other (Coulomb potential). However, at distances smaller than a nuclear radius, the potential cannot be the Coulomb one (repels nuclei), but it must be attractive: short-range nuclear forces overcome electrostatic forces. At $r = 0$, this potential should be $\sim -30\,\text{MeV}$ (from the mass deficiency), and at around the nuclear radius ($r \approx 1\,\text{fm}$), it should become the Coulomb potential, $\sim +2e^2/r$.

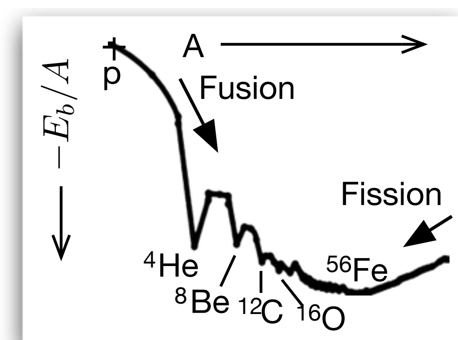
Temperature for fusion (quantum tunnelling): considering for a proton $p = h/\lambda$, and $r_{\text{action}} \rightarrow \lambda$, we get kinetic energy $= \dfrac{1}{2}\mu v^2 = \dfrac{p^2}{2\mu} = \dfrac{h^2}{2\mu\lambda^2}$, where $\mu = m_p^2/(2m_p) = m_p/2$ is the reduced mass (two body problem). This has to be equal to Coulomb barrier $\sim \dfrac{2e^2}{\lambda}$, so we solve for the wavelength $\lambda \sim \dfrac{h^2}{2e^2\mu}$. Then, by the equipartition theorem, $\dfrac{3}{2}k_B T = \dfrac{h^2}{2\mu\lambda^2} \implies T \sim 10^7\,\text{K}$, which is the same order of magnitude of the temperature of the Sun we estimated in § Order of magn. approx.

CNO cycle: another way to fusion hydrogen to helium, but, using carbon, nitrogen and oxygen as catalysts (i.e., the fusion of H with isotopes of N, C and O [+decays] releases helium at some point). It requires the previous presence of N, C and O.

Binding energy: mass-energy difference between isolated and bound components: $E_b = [\bar{Z}m_p + (\bar{A} - \bar{Z})m_n - m_{\text{nuc}}]c^2$, where $\bar{Z}$ is the number of protons, $\bar{A}$ is the number of nucleons and m_{nuc} is the experimental value of the mass of the bound nucleus. The values are plotted in the figure (seeing it reflected helps a lot!). For elements to the left of iron, fusion with a proton is energetically favorable and the processes are exothermic (except for ^{4}He, ^{12}C, ^{16}O, that are stable). For elements heavier than iron, energy has to be invested to create elements even heavier, and nuclear fission (radioactive decay) releases energy when heavy elements turn into lighter ones.

Alpha process/ladder: when H is exhausted from a star, He starts burning and fusing into heavier elements (alpha particle = He nucleus). This series of reactions creates heavier and heavier elements until Ni and Fe are reached.





Stellar evolution

Main seq., Sun-like to high mass stars ($M \gtrsim 1.2 M_\odot$): they follow mainly the CNO cycle, which requires high temperatures (lower mass stars follow mainly the PP chain). Radiation cannot transport energy fast enough in the core, so, a temperature gradient builds up and induces convection in the core $\Longrightarrow$ convective core. Massive stars have radiative envelopes, but in the most massive stars, the core is so big that they can become fully convective.

Later phases, low mass stars ($M \lesssim 8 M_\odot$):

Hydrogen shell burning: when the hydrogen is exhausted, the star core composed of helium surrounded by a shell of hydrogen. Temperatures and pressures can be high enough to cause hydrogen fusion in the hydrogen shell (and higher than in the main sequence). During this process, some of this extra energy expands the envelope, so it cools down and the star becomes luminous, large and cool: a red giant.

Helium core burning: after that, the helium core begins to collapse until $\rho \sim 10^7\,\mathrm{kg/m^3}$, $T \sim 10^8\,\mathrm{K}$, when helium starts burning according to

$$^4_2\mathrm{He} + ^4_2\mathrm{He} \rightleftharpoons ^8_4\mathrm{Be}$$
$$^8_4\mathrm{Be} + ^4_2\mathrm{He} \rightarrow ^{12}_6\mathrm{C}$$

(triple alpha process: alpha + (2 alpha → Be) → C)

Be-8 decays quickly, so it has to react quickly with another alpha particle to produce C-12. Then, C-12 can also fuse with another alpha particle to produce O-16.

Helium shell burning and planetary nebulae: After that, the star burns hydrogen in an outer shell, burns He in an inner shell, and contains a C/O core. During this phase, the high energy inside not only expands, but expels parts of the atmosphere (stellar winds) at rates of $\dot{M} \sim 10^{-4}\,\mathrm{M_\odot/yr} \Longrightarrow$ most of the mass is lost after 10 000 yr, and forms a planetary nebula that expands at rates of $\sim [10{,}30]\,\mathrm{km/s}$.

Time scales for a solar-mass star: H-core burning (main sequence): $\sim 10^{10}$ yr. H-shell burning (red giant): $\sim 10^9$ yr. He-core burning (horizontal branch): $\sim 10^8$ yr. He-shell burning (asymptotic giant branch): $\sim 10^7$ yr. Planetary nebula formation: $\sim 10^4$ yr. End result: white dwarf.

Later phases, massive stars ($M \gtrsim 8M_\odot$): all previous phases minus planetary nebulae. Instead, a set of reactions called collectively silicon burning takes place.

Carbon-alpha-oxygen burning ($T \gtrsim 10^8$ K):

$$^{12}_{6}C + ^{4}_{2}He \rightarrow ^{16}_{8}O + \gamma$$
$$^{16}_{8}O + ^{4}_{2}He \rightarrow ^{20}_{10}Ne + \gamma$$

(C + alpha → O; O + alpha → Ne)

Carbon-carbon burning ($T \gtrsim 6 \cdot 10^8$ K):

$$^{12}_{6}C + ^{12}_{6}C \rightarrow \begin{cases} ^{16}_{8}O + 2\,^{4}_{2}He, & ^{23}_{12}Mg + n \\ ^{20}_{10}Ne + 2\,^{4}_{2}He, & ^{24}_{12}Mg + \gamma \\ ^{23}_{11}Na + p \end{cases}$$

Oxygen-oxygen burning ($T \gtrsim 10^9$ K):

$$^{16}_{8}O + ^{16}_{8}O \rightarrow \begin{cases} ^{24}_{12}Mg + 2\,^{4}_{2}He, & ^{31}_{16}S + n \\ ^{28}_{14}Si + 2\,^{4}_{2}He, & ^{32}_{16}S + \gamma \\ ^{31}_{15}P + p \end{cases}$$

Silicon-alpha-burning ($T \gtrsim 3 \cdot 10^9$ K)

$$^{28}_{14}Si + ^{4}_{2}He \rightleftharpoons ^{32}_{16}S + \gamma$$
$$^{32}_{16}S + ^{4}_{2}He \rightleftharpoons ^{36}_{18}Ar + \gamma$$

(Si + alpha → S; S + alpha → Ar, etc.)

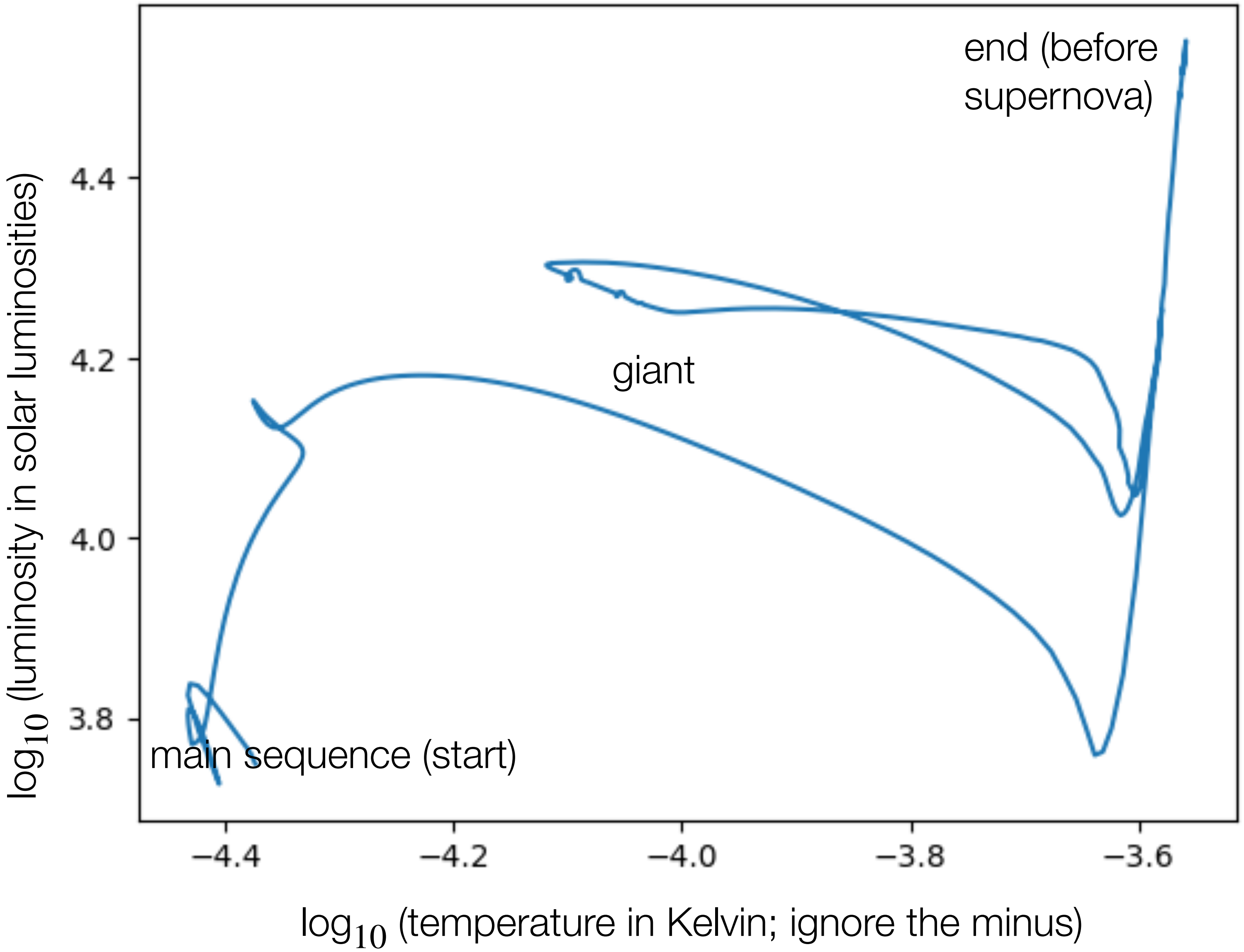
... until

$$^{48}_{24}Cr + ^{4}_{2}He \rightleftharpoons ^{52}_{26}Fe + \gamma$$
$$^{52}_{26}Fe + ^{4}_{2}He \rightleftharpoons ^{56}_{28}Ni + \gamma$$

(Cr + alpha → Fe; Fe + alpha → Ni)

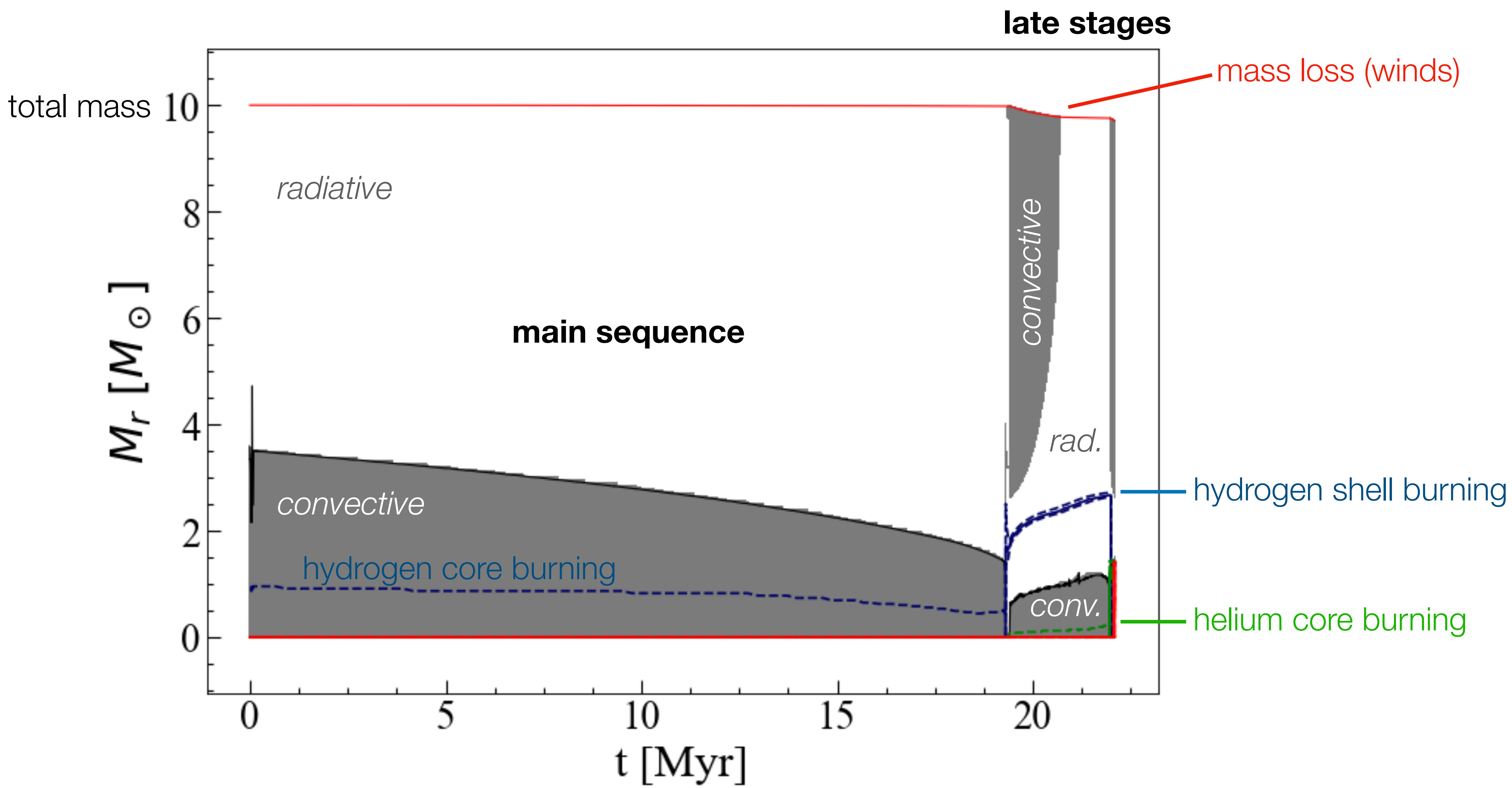
where the process becomes endothermic. Photons possess enough energy to destroy heavy nuclei; this is photodisintegration (e.g., Fe + γ → 13 alpha + 4 n, or alpha → 2 p + 2 n).

Timescales for a $25M_\odot$ star: H-core burning: $5 \cdot 10^6$ yr; He-core burning: $5 \cdot 10^5$ yr; C-core burning: 500 yr; Si-burning: 1 day. After that: (type II) supernova explosion, compact object.



HR diagram of a massive star

This plot is the HR diagram of a 10 solar mass star computed with the Geneva stellar evolution code (Genec). The main sequence evolution lasts longer than the other phases, which is not evident from this plot. The stellar radius can be easily computed with the Stefan-Boltzmann law as $R_\star = \sqrt{L_\star/(4\pi\sigma_{\text{Stef-Boltz}})}T^2_{\text{surf}}$



Kippenhahn diagram of a massive star

This diagram summarizes the structure and evolution of a massive star of 10 solar masses (as computed with Genec). As seen previously, the main sequence has a convective core and a radiative envelope. The structure changes when hydrogen burning moves from the core to the shell and helium core burning starts. At the end of the life of a star, mass loss due to winds becomes significant.

Supernovae

Timescale for core collapse: from the hydrostatic equilibrium analysis, we get for the non-equilibrium case, $dM\frac{d^2r}{dt^2} = -dP - \frac{GM}{r^2}dM$, where the acceleration is only radial. If we turn the pressure off suddenly, we get a free fall collapse. Then, $\frac{d^2r}{dt^2} = -\frac{GM}{r^2}$. Now, we can approximate crudely, $\frac{R}{\tau_{ff}^2} \sim \frac{GM}{R^2}$. For a stellar core with $M \approx M_\odot$ and $R \approx 6 \cdot 10^6$ m (core ~ size of the Earth), $\tau_{ff} \sim 1$ s.

Energy released: during the collapse of the core, the gravitational energy released is approx. the difference between the initial and final states of the star: $U_{before} = -GM_\odot^2/(6 \cdot 10^6\,\text{m}) \sim -4 \cdot 10^{43}$ J, and $U_{after} = -GM_\odot^2/(10^4\,\text{m}) \sim -4 \cdot 10^{43}$ J $\implies \Delta U \sim 10^{46}$J, most of which is carried out by the neutrinos of the inverse beta decay during neutronization. The kinetic energy released is of the order of 10^{44} J, and the energy carried out by photons, of the order of 10^{42} J. The peak brightness of a supernova is of around $10^9\,L_\odot$.

Elements beyond iron: in a neutron-rich environment, an atomic nucleus can capture a neutron; this neutron then undergoes beta decay (the captured neutron becomes a proton and releases an electron + neutrino), leaving the nucleus with 1 proton more than before. There are two processes that involve these reactions: if neutrons are captured more rapidly than beta decay can eliminate them, heavy neutron-rich elements build up rapidly (*r-process*); on the contrary, heavy elements build up slowly (*s-process*).

Supernova classification: based on spectral lines, not physical mechanisms. Basic: type I (with hydrogen) and type II (no hydrogen). Type Ia supernovas are caused by the detonation of a white dwarf by accretion (from a donor star). The rest (type Ib, Ic, II) are core-collapse supernovas.

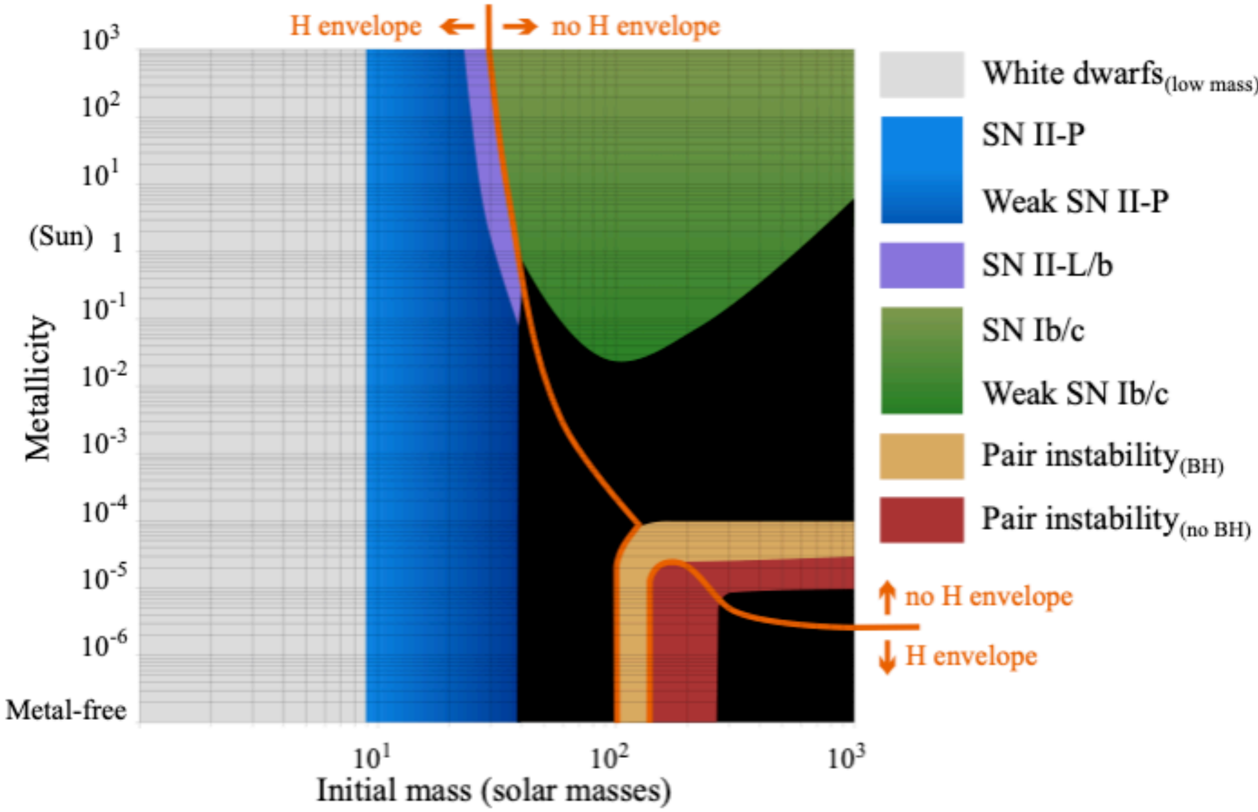
Type Ia: the core mass of a white dwarf exceeds the Chandrasekhar limit because of accretion from a donor star. All type I supernovae share a similar rate of decline of their brightness after their peak. This can be explained by the half-life of β decay from $^{56}_{27}Co$ to $^{56}_{27}Fe$ ($\tau = 77.7$ d). The total number of atoms present with time is $N(t) = N_0 e^{-\ln(2)t/\tau}$. From the energy given to the supernova by the reaction, it can be estimated that the magnitude (or change or the logarithm of luminosity with time) decays at a rate of ~ 0.01 mag/d, the rate that is observed after 50 days in a type Ia supernova. This is why they are used as standard candles.

Type II: isolated stars. Highly dependent on the mass of the star (mass-energy available) and metallicity (how effective is radiation pressure: more metals means more opaque media where radiation pressure produces more force). Photodesintegration of nuclei by radiation and neutronization reduce radiation support at the core of the star, rapid triggering gravitational core collapse (see estimation of timescale above $\Longrightarrow$ speeds of up to $70\,000\,\mathrm{km/s}$ for the outer core). The outer core collapses supersonically, but the inner core collapses subsonically. When nuclear densities ($\sim 10^{14}\,\mathrm{g/cm^3}$) are reached at the inner core, the nuclear force (residual strong interaction) repels the infalling material, halting the collapse.

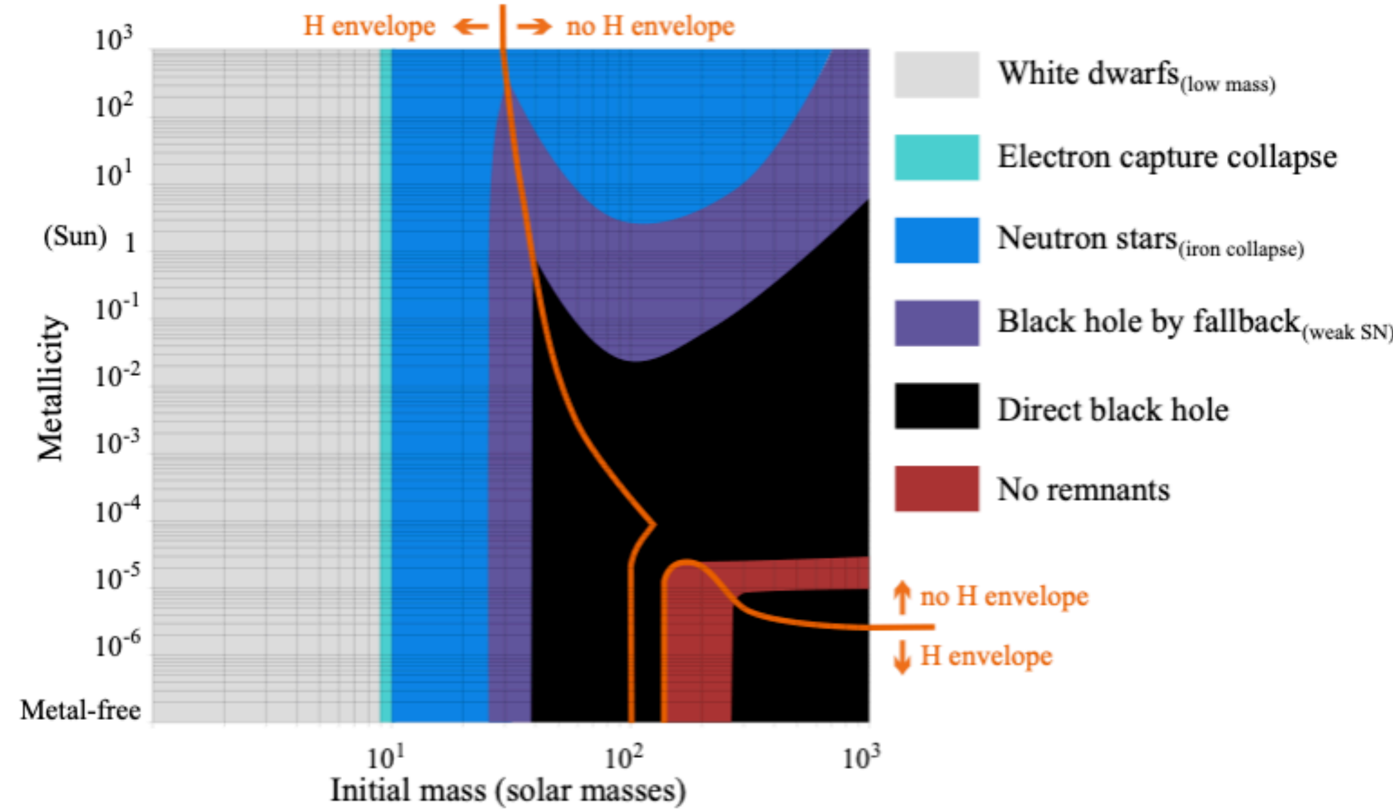
Type II: rebound: Then, the outer material rebounds and a pressure shock wave propagates outwards. The material ejection process is not fully understood, but it could involve neutrinos being trapped by the fast increase of density in the inner core. Neutrinos diffuse outwards through the outer core and their interactions (neutrino transport) may push the material outwards until they reach the "neutrinosphere" (low density threshold), beyond which they can escape without further interaction. At the boundary of the outer core, there is still a Si burning shell.

Pair-creation instability: in very massive cores, gamma rays can create electron-positron pairs instead of being used for radiation pressure. This lowers radiation pressure and triggers collapse. Then, oxygen is ignited explosively and mass is lost from the core, even leaving no remnant behind. For this massive cores, it is assumed the mass lost by stellar winds during the main sequence is small, which means low metallicity ($\Longrightarrow$low opacity $\Longrightarrow$low radiation pressure that drives the wind).

Fate of the most massive stars: in the diagram of the supernova types, for the most massive stars in the less-than-solar metallicity regime, the direct formation of a black hole means that no supernova shock is launched, and so, there is only very faint or no supernova explosion observed.



Supernova types. The black regions mean no SN shock is launched.

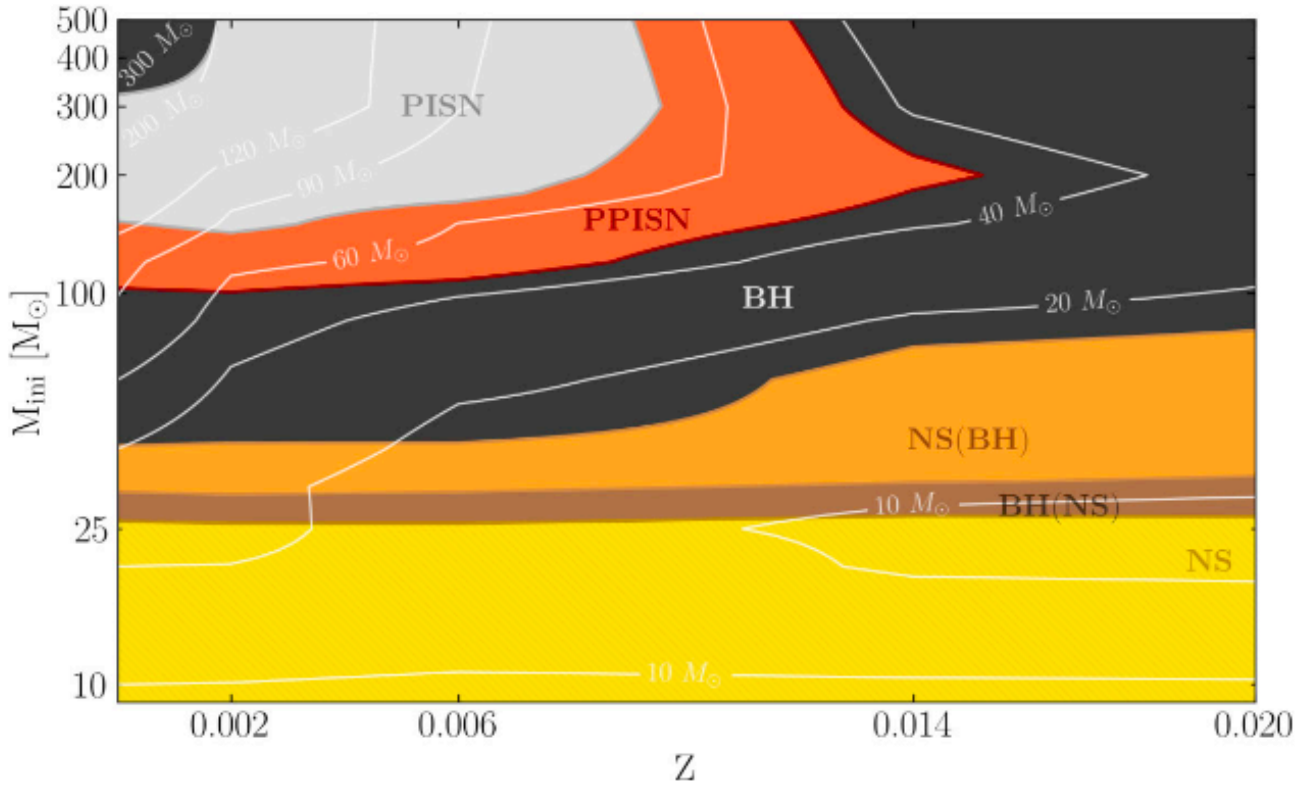


Remnants for a single progenitor. (Adapted from Wikipedia)

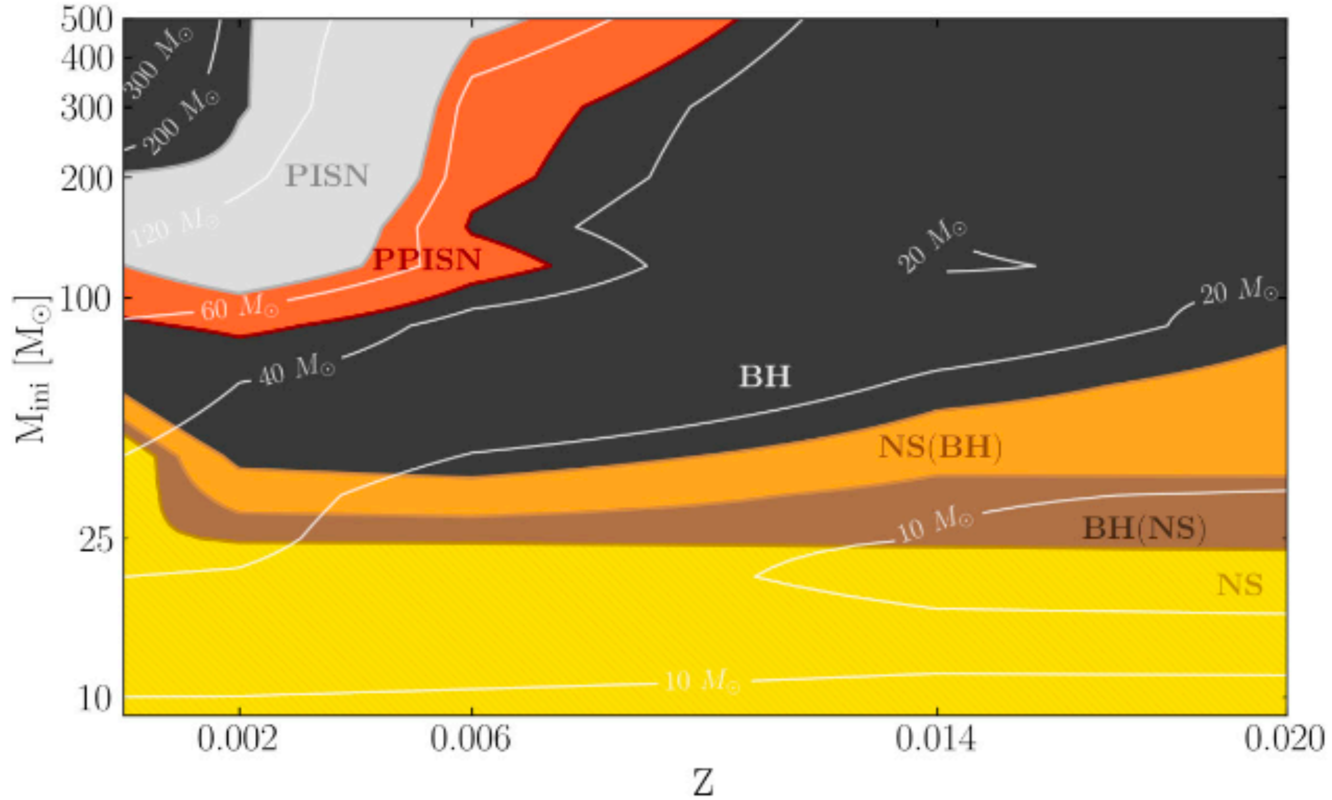
Supernova types and remnants

Not all supernovae are the same. These plots show the supernova types for isolated stars. As it can be seen from the plot, the kind of supernova and the kind of remnant produced depend strongly on metallicity. For low metallicity, direct collapse onto blackholes becomes more common compared to solar-like metallicity.

Heger et al. (2003) ApJ 591 288.



Remnants for non-rotating models



Remnants for rotating models

Remnants for rotating stars

The classical description of supernovae and remnants does not include rotation. In this paper, the plot from the previous page has been updated to include rotation. Rotating models show that at high metallicities it's more difficult to have pair-instability supernovae.

Hirschi et al 2025 MNRAS 543 2796

Stellar remnants

Introduction to general relativity

Space-time is described by the variables $x^0 = ct$, x^1, x^2, x^3 (e.g., $= x, y, z$ in Cartesian coordinates). In Euclidian space, we can define the distance between two points by using the Pythagorean theorem: $ds^2_{\text{Eucl}} = dx^2 + dy^2 + dz^2$. In *special* relativity, however, we know that this distance is not measured the same by an observer moving with respect to us, due to Lorentz contraction. The closest we can define to an invariant "distance" is called *spacetime interval*: $ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$ (this can be shown to be invariant for any observer by using the Lorentz transformations, differentiating and substituting). The meaning of the spacetime interval can be seen by placing ourselves in a frame of reference of a moving particle. In that frame of reference, the particle appears to be static ($dx = dy = dz = 0$), and the measured time is the proper time $d\tau$, so that $ds^2 = -c^2d\tau^2$. Since ds^2 is an invariant, we can always write that relation in any frame.

In general relativity, the spacetime interval also shows the curvature of the spacetime, which is caused by the presence of matter and energy. In order to start the derivation, we are interested in knowing the spacetime interval (sometimes called "the metric") outside of a simple spherical object of mass M. The full solution is called the Schwarzschild solution. As a first order approximation (enough for this derivation), a comparison with Newtonian gravity yields

$$ds^2 \approx -\left(1 + \frac{2\Phi}{c^2}\right) c^2dt^2 + dx^2 + dy^2 + dz^2$$

(this comparison can be done by studying the motion of a particle under a weak potential in classical mechanics, with a Lagrangian, and relativity, by saying that the trajectory of a particle has to minimize the proper time [variational calculus]). For a spherical object of mass M the potential is $\Phi = -GM/r$.

The exact metric for the exterior spacetime that describes a spherically-symmetric object in spherical-like coordinates is

$$ds^2 = -\left(1 - \frac{2GM}{c^2r}\right) c^2dt^2 + \left(1 - \frac{2GM}{c^2r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

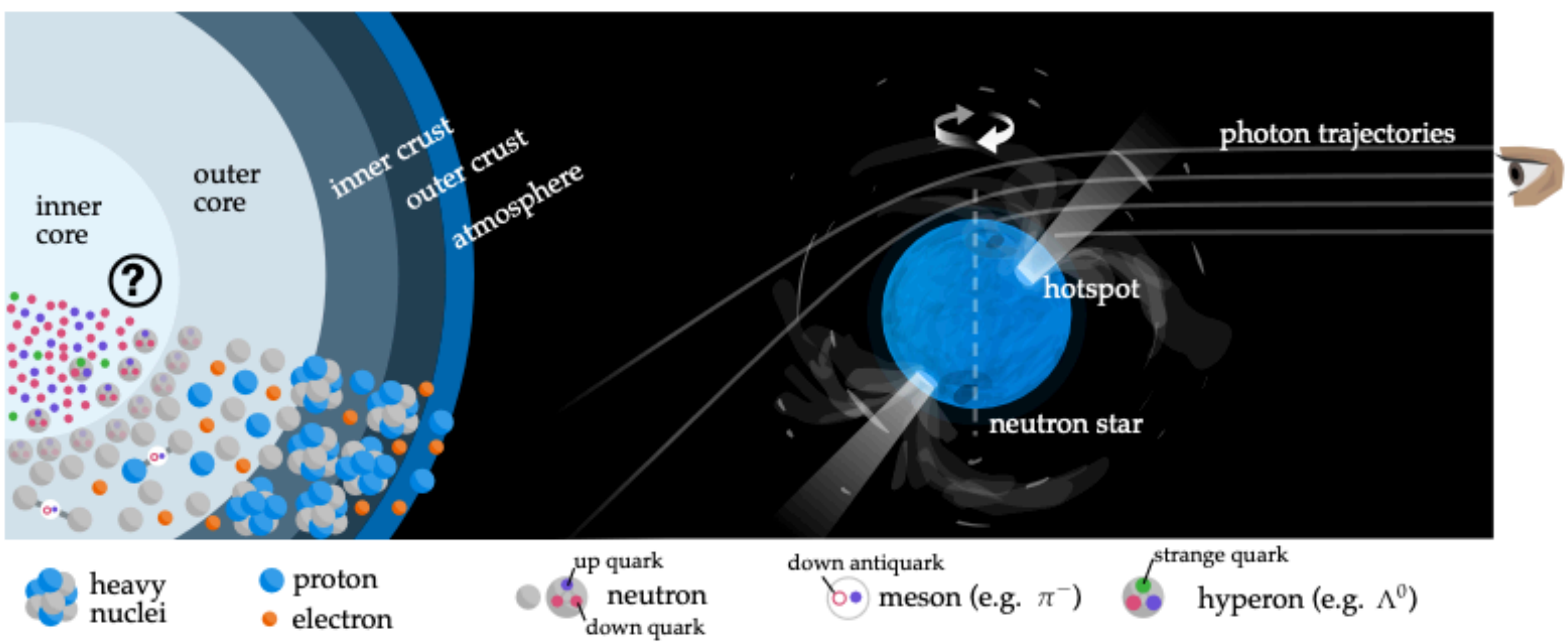
and it is called the *Schwarzschild metric*. This metric is used for example for understanding **black holes**, regions of spacetime so distorted that light cannot escape from there.

Neutron stars: introduction to the equation of state:

After the supernova explosion, the stellar core collapses under gravity and squeezes together. Under those extreme circumstances, *inverse beta decay* starts to be more and more energetically favorable to the heavy nucleons in the stellar core, according to the equation

$$\text{mass-energy} + p + e \longrightarrow n + \nu.$$

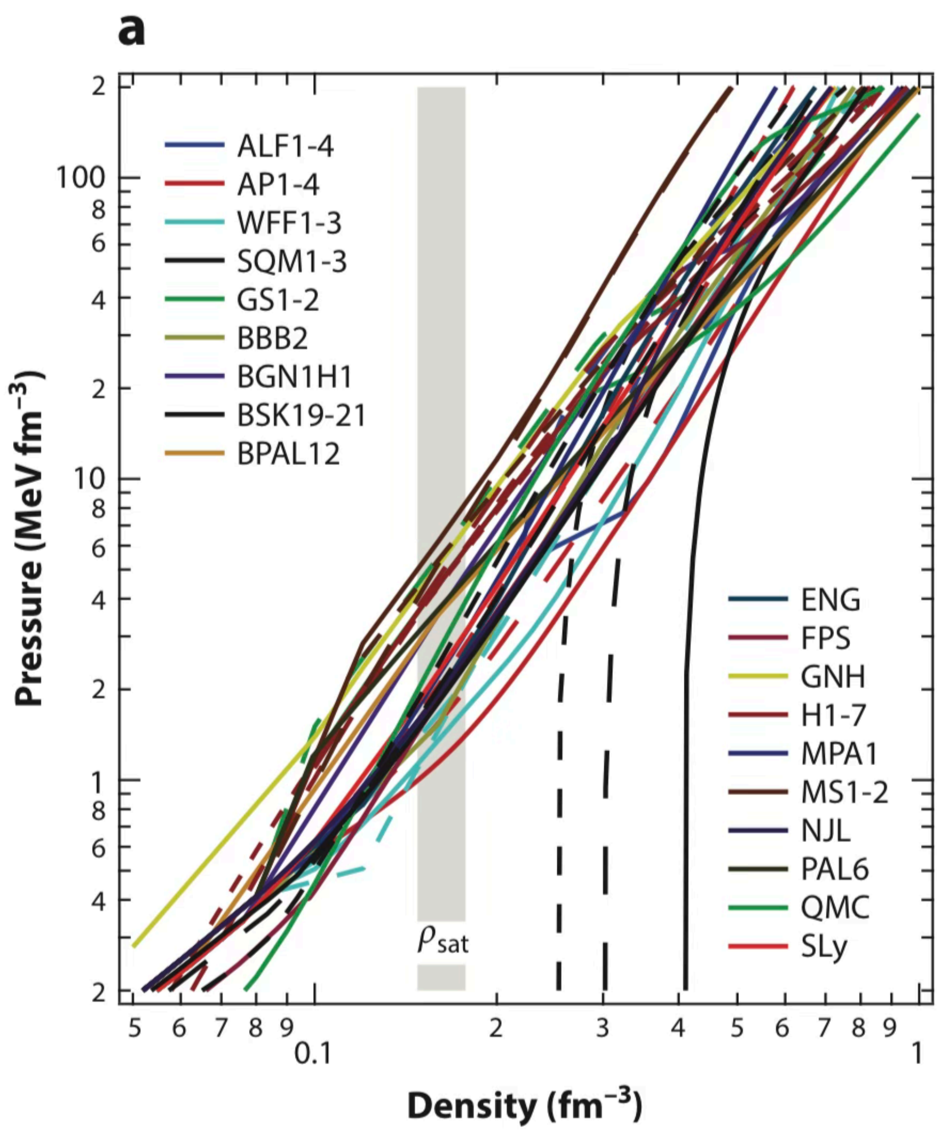
Normally, the inverse equation happens (beta decay), meaning, neutrons on their own decay into protons, electrons, and neutrinos. Under the inverse beta decay, matter becomes *neutronized* and this process is called *neutron drip*. A neutron star is not composed entirely of neutrons, but instead several degrees of neutronization happen in different layers of the neutron stars (there is an electron atmosphere, a crust, a core). We don't fully know what the core is made from: this is the problem of the equation of state.

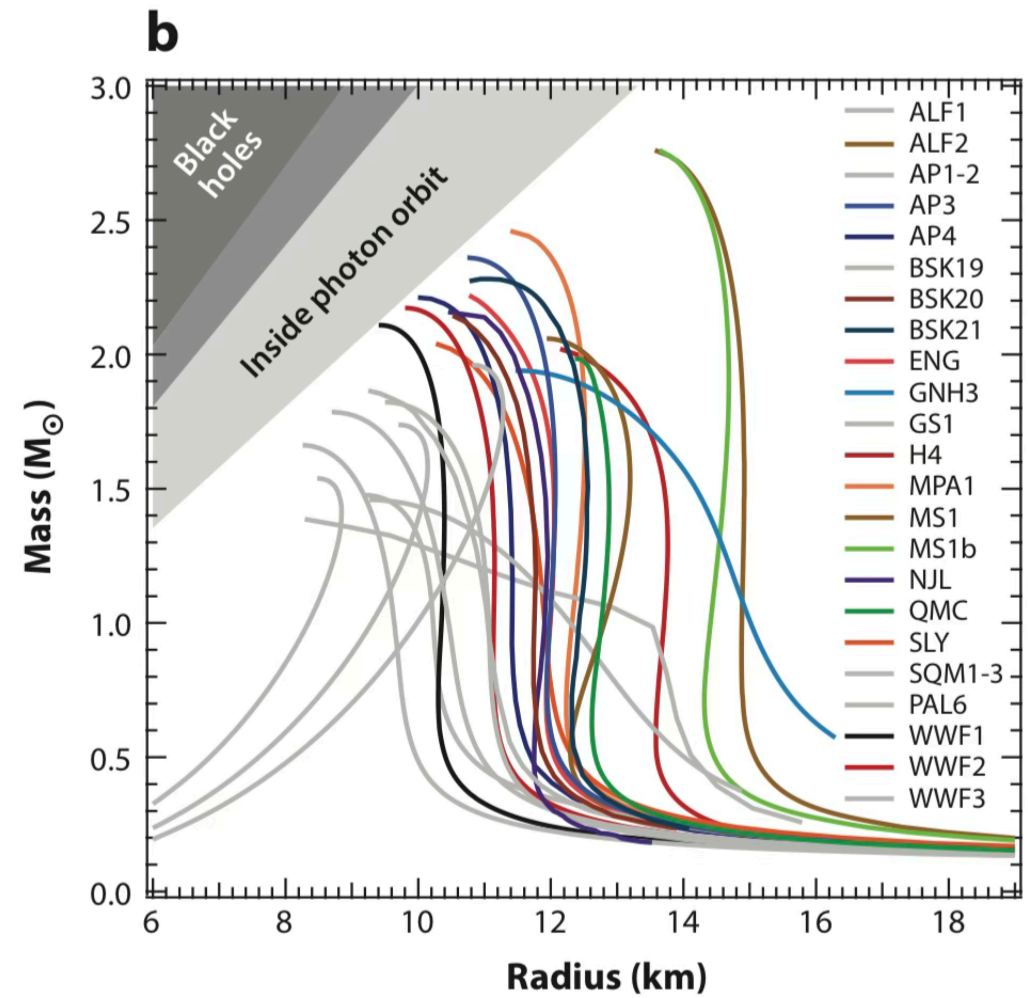


Neutron stars: approximation of the degenerate gas: an order-of-magnitude argument to derive the degeneracy pressure that happens inside of a neutron star is as follows. From quantum mechanics, we know: phase-space cells are discretized ($\Delta p \Delta x \sim \hbar$), and two identical fermions cannot have the same quantum state simultaneously. There is a minimum energy that a group of N fermions can have so they don't have the same quantum state simultaneously. That minimum (kinetic) energy (Fermi energy) determines a Fermi momentum, which originates the pressure. Then, the pressure should depend on the number of fermions, and the phase-space cell size. We can include the spatial size in a volume, and say that P should depend on n, the number density. The phase-space size is also dependent on mass (more mass, more momentum), and $\hbar$ comes into play because of quantization of the phase space. The pressure should not depend on temperature, since the pressure must be felt in the minimum energy state ("$T = 0$"). Then, $P \sim \hbar^{\alpha} m^{\beta} n^{\gamma}$, where the exponents are determined by dimensional analysis, yielding $P \sim \hbar^2 m^{-1} n^{5/3}$, in agreement with the correct calculation (except for a dimensionless constant of order unity). For a relativistic degenerate gas, we replace the dependency $m \to c$ and recalculate exponents so we get $P \sim \hbar c n^{4/3}$.

Mass and radius: if we solve the relativistic equivalent to the structure equations (TOV equations) for a neutron star, we find out that depending on what we assume the neutron star is composed of, we get different radii for a given neutron star mass. Strangely, more massive neutron stars tend to be *smaller,* not larger (they become more compact). If we can measure the mass and radius of neutrons stars very accurately, we could find out what they are made of, from the predictions of the radius by equations of state.

Pulsars: when matter falls down onto the surface of a neutron star, its gravitational energy is released as radiation. Because of the strong magnetic fields of neutron stars, matter is accreted near magnetic poles on the surface magnetic field. Those sites become *hotspots*. As the neutron star rotates and the light from a hotspot faces us on Earth, we see a maximum of radiation, and when a hotspot is on the hidden side of the neutron star, we see a minimum. This is a pulsar. Pulsars can rotate at incredibly fast speeds, reching up to 716 Hz for the fastest spining pulsar known to date. Because spacetime is curved, and light curves with it, we actually can see light from behind the neutron star in front. The pulses of a pulsar can be used to measure the radii of neutron stars.





Equation of state and mass-radius curve

The left plot shows several calculations of the equation of state of a neutron star assuming different compositions. The right plot shows what the predicted masses and radii yield for different compositions. Except for the few gray curves, more massive a neutron star is, the smaller the radius.

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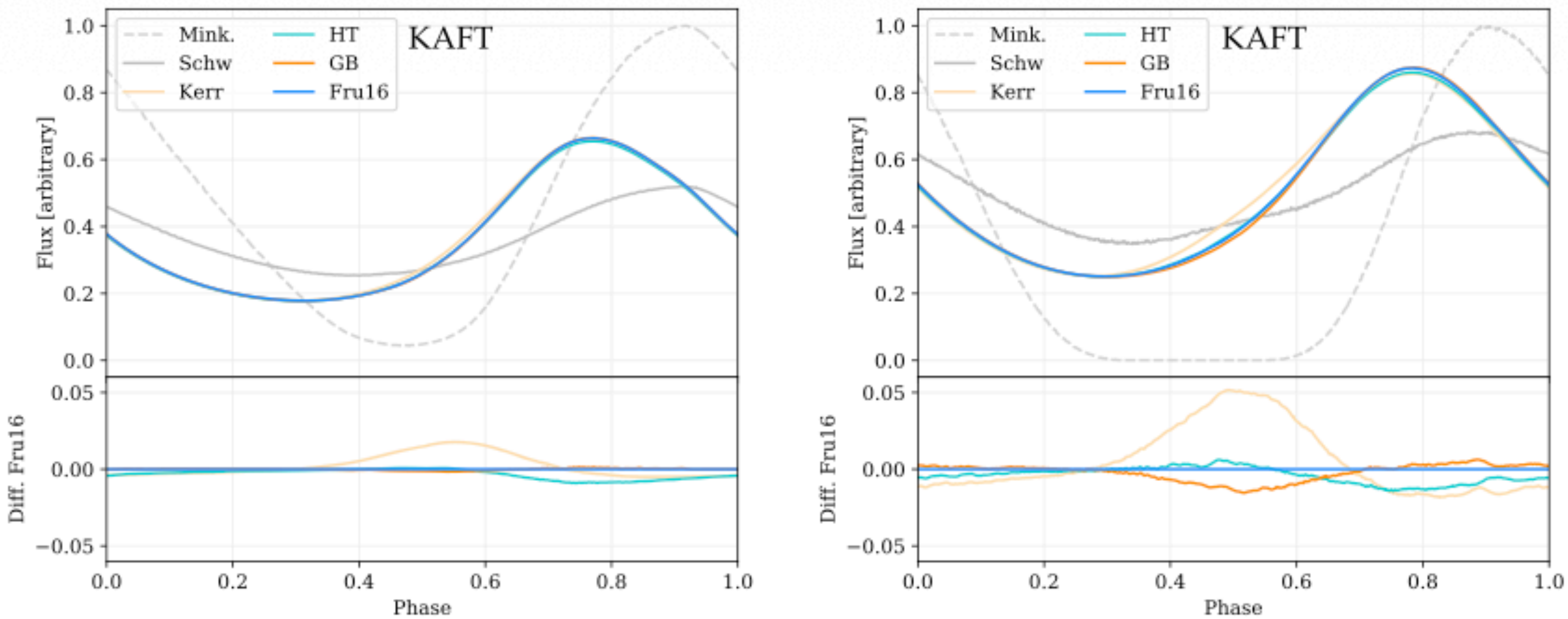

(a) (b)

View: $i = 30°$; *hot spot:* circular $\Theta = 45°$ $\zeta = 10°$

View: $i = 0°$; *hot spot:* crescent
$\Theta_c = 2.8394\,\mathrm{rad}$ $\zeta_c = 0.46\,\mathrm{rad}$ $\phi_{c0} = 0.0606\,\mathrm{rad}$
$\Theta_s = 2.91\,\mathrm{rad}$ $\zeta_s = 0.47\,\mathrm{rad}$ $\phi_{s0} = 0\,\mathrm{rad}$

Pulses of a pulsar

These plots show the pulses of a pulsar assuming different models for the spacetime (different line colors). The left and right panel show different shapes and locations of the hotspots in the neutron star.

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