



Compact objects

Introduction to general relativity

Space-time is described by the variables $x^0 = ct$, x^1, x^2, x^3 (e.g., $= x, y, z$ in Cartesian coordinates). In Euclidian space, we can define the distance between two points by using the Pythagorean theorem: $ds_{\text{Eucl}}^2 = dx^2 + dy^2 + dz^2$. In *special* relativity, however, we know that this distance is not measured the same by an observer moving with respect to us, due to Lorentz contraction. The closest we can define to an invariant "distance" is called *spacetime interval*: $ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$ (this can be shown to be invariant for any observer by using the Lorentz transformations, differentiating and substituting). The meaning of the spacetime interval can be seen by placing ourselves in a frame of reference of a moving particle. In that frame of reference, the particle appears to be static ($dx = dy = dz = 0$), and the measured time is the proper time $d\tau$, so that $ds^2 = -c^2 d\tau^2$. Since ds^2 is an invariant, we can always write that relation in any frame. Another way of interpreting ds is the difference between the square of the spatial distance as an object moves and the distance travelled by a light signal in the same time. This means that for light, $ds^2 = 0$.

In general relativity, the spacetime interval also shows the curvature of the spacetime, which is caused by the presence of matter and energy. In order to start the

derivation, we are interested in knowing the spacetime interval (sometimes called "the metric") outside of a simple spherical object of mass M . The full solution is called the Schwarzschild solution. As a first order approximation (enough for this derivation), a comparison with Newtonian gravity yields

$$ds^2 \approx - \left(1 + \frac{2\Phi}{c^2} \right) c^2 dt^2 + dx^2 + dy^2 + dz^2$$

(this comparison can be done by studying the motion of a particle under a weak potential in classical mechanics, with a Lagrangian, and relativity, by saying that the trajectory of a particle has to minimize the proper time [variational calculus]). For a spherical object of mass M the potential is $\Phi = -GM/r$.

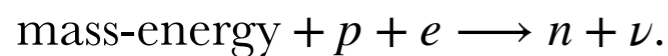
The exact metric for the exterior spacetime that describes a spherically-symmetric object in spherical-like coordinates is

$$ds^2 = - \left(1 - \frac{2GM}{c^2 r} \right) c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r} \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

and it is called the *Schwarzschild metric*. This metric is used for example for understanding **black holes**, regions of spacetime so distorted that light cannot escape from there.

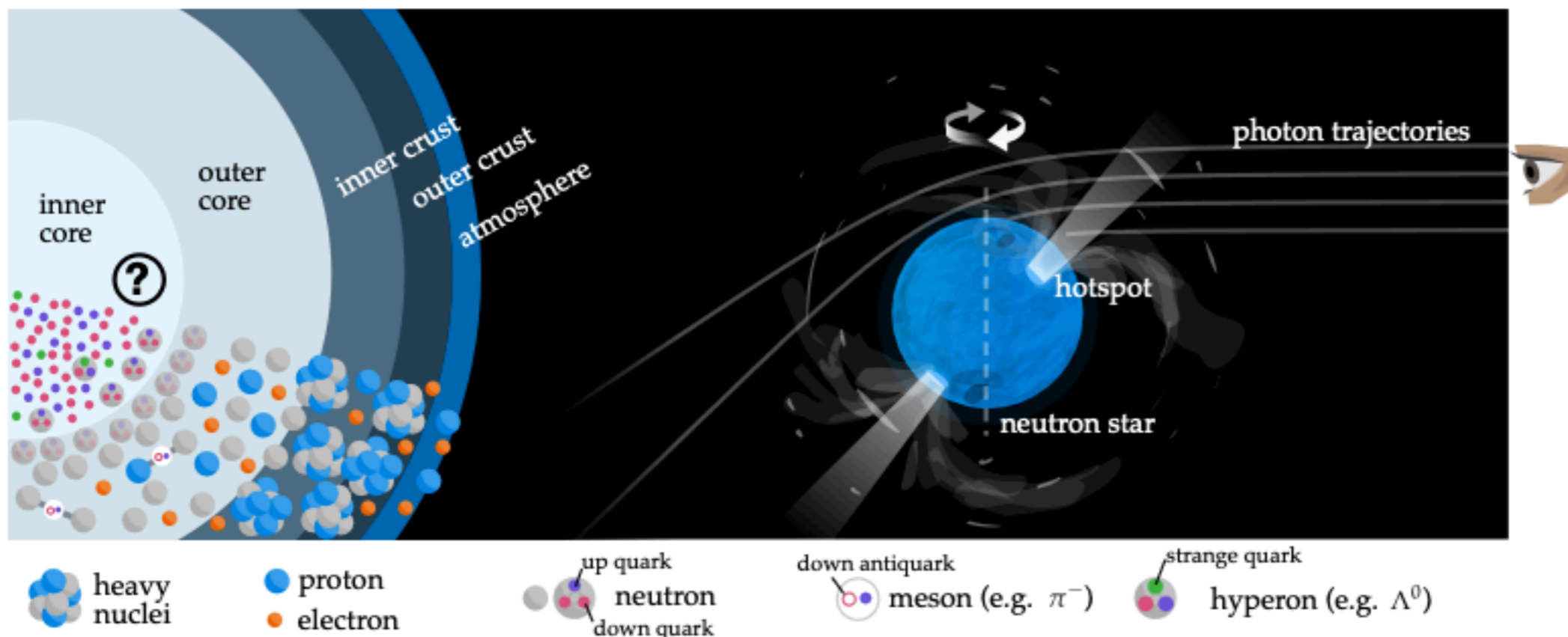
Neutron stars: introduction to the equation of state:

After the supernova explosion, the stellar core collapses under gravity and squeezes together. Under those extreme circumstances, *inverse beta decay* starts to be more and more energetically favorable to the heavy nucleons in the stellar core, according to the equation



Normally, the inverse equation happens (beta decay), meaning, neutrons on their own decay into protons, electrons, and neutrinos. Under the

inverse beta decay, matter becomes *neutronized* and this process is called *neutron drip*. A neutron star is not composed entirely of neutrons, but instead several degrees of neutronization happen in different layers of the neutron stars (there is an electron atmosphere, a crust, a core). We don't fully know what the core is made from: this is the problem of the equation of state.



Neutron stars: approximation of the degenerate gas: an order-of-magnitude argument to derive the degeneracy pressure that happens inside of a neutron star is as follows. From quantum mechanics, we know: phase-space cells are discretized ($\Delta p \Delta x \sim \hbar$), and two identical fermions cannot have the same quantum state simultaneously. There is a minimum energy that a group of N fermions can have so they don't have the same quantum state simultaneously. That minimum (kinetic) energy (Fermi energy) determines a Fermi momentum, which originates the pressure. Then, the pressure should depend on the number of fermions, and the phase-space cell size. We can include the spatial size in a volume, and say that P should depend on n , the number density. The phase-space size is also dependent on mass (more mass, more momentum), and \hbar comes into play because of quantization of the phase space. The pressure should not depend on temperature, since the pressure must be felt in the minimum energy state (“ $T = 0$ ”). Then, $P \sim \hbar^\alpha m^\beta n^\gamma$, where the exponents are determined by dimensional analysis, yielding $P \sim \hbar^2 m^{-1} n^{5/3}$, in agreement with the correct calculation (except for a dimensionless constant of order unity). For a relativistic degenerate gas, we replace the dependency $m \rightarrow c$ and recalculate exponents so we get $P \sim \hbar c n^{4/3}$.

Mass and radius: if we solve the relativistic equivalent to the structure equations (TOV equations) for a neutron star, we find out that depending on what we assume the neutron star is composed of, we get different radii for a given neutron star mass. Strangely, more massive neutron stars tend to be *smaller*, not larger (they become more compact). If we can measure the mass and radius of neutrons stars very accurately, we could find out what they are made of, from the predictions of the radius by equations of state.

Pulsars: when matter falls down onto the surface of a neutron star, its gravitational energy is released as radiation. Because of the strong magnetic fields of neutron stars, matter is accreted near magnetic poles on the surface magnetic field. Those sites become *hotspots*. As the neutron star rotates and the light from a hotspot faces us on Earth, we see a maximum of radiation, and when a hotspot is on the hidden side of the neutron star, we see a minimum. This is a pulsar. Pulsars can rotate at incredibly fast speeds, reaching up to 716 Hz for the fastest spinning pulsar known to date. Because spacetime is curved, and light curves with it, we actually can see light from behind the neutron star in front. The pulses of a pulsar can be used to measure the radii of neutron stars.

Einstein Field Equations: in order to find the metric (shape) of the spacetime, we use the Einstein Field equations. In this course, we only care about their meaning. Schematically, they can be expressed as

$$\left(\begin{array}{c} \text{curva-} \\ \text{ture of} \end{array} \right) (\text{spacetime}) = \left(\begin{array}{c} \text{matter} \\ \text{energy} \end{array} \right)$$

that is, matter-energy acts as a source of the curvature of spacetime. The curvature "operator" contains derivatives of the metric (here, the "spacetime"). For example, if one has a spherically-symmetric distribution of matter, one can obtain the Schwarzschild metric as a result.

Geometrized units: if we set $G = c = 1$, we create *geometrized units*. Length, time and mass are then all measured in units of length.

General relativistic stellar structure equations (TOV equation): assume you have a metric described by the interval

$$ds^2 = - e^{2\Phi(r)} dt^2 + e^{2\Lambda(r)} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)$$

This represents the most general solution of the Einstein Field Equations with spherical symmetry. If you insert this metric in the left hand side of the field equations and then a spherical fluid in the right hand side, one obtains the following set of equations:

$$\mathbf{a)} \quad e^{-2\Lambda} = 1 - \frac{2m}{r} \quad (\text{radial part of the metric})$$

$$\mathbf{b)} \quad \frac{d\Phi}{dr} = \frac{m + 4\pi r^3 p}{r(r - 2m)} \quad (\text{time part of the metric})$$

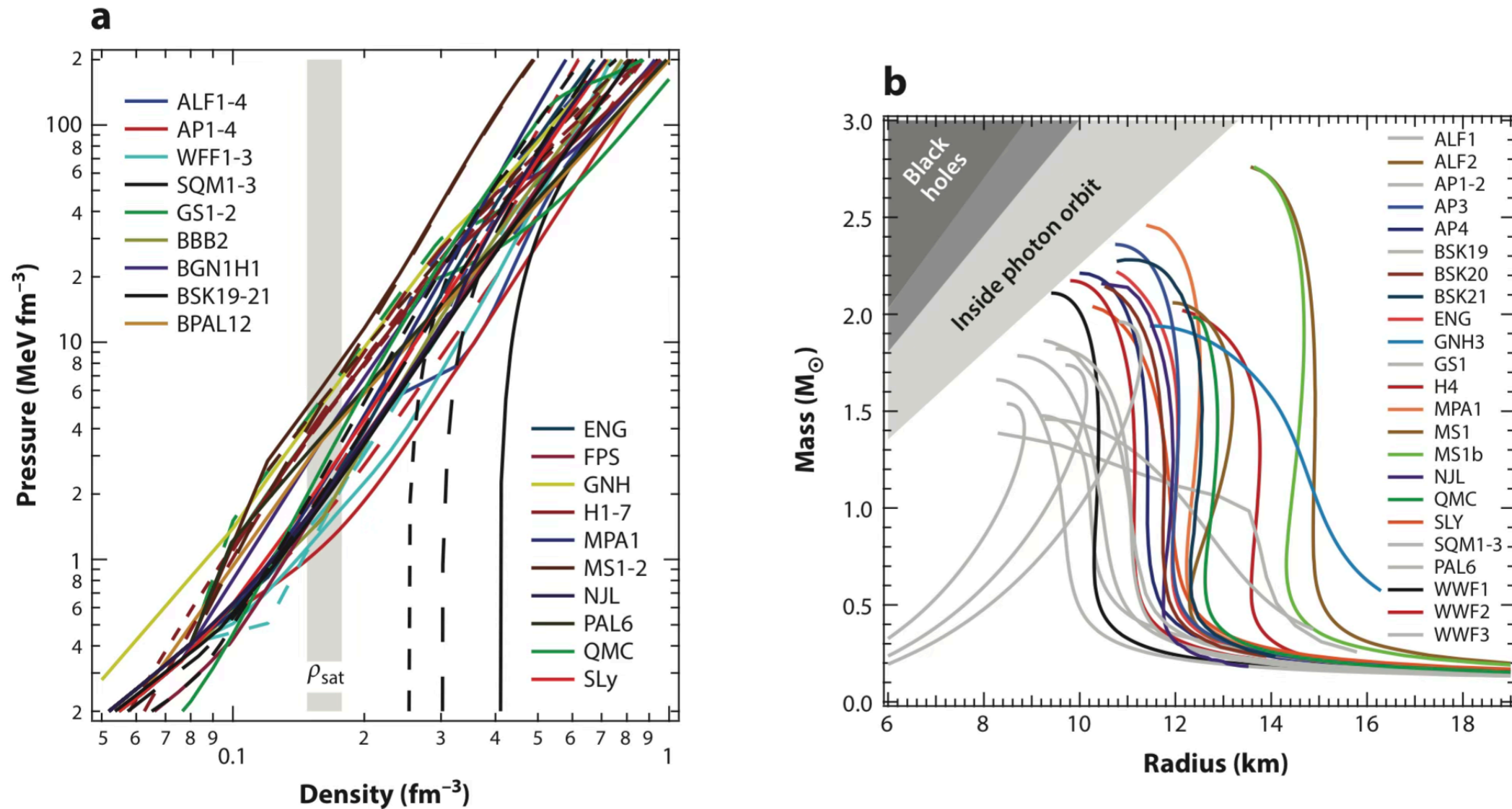
$$\mathbf{c)} \quad \frac{dp}{dr} + (p + \rho) \frac{d\Phi}{dr} = 0 \quad (\text{hydrostatic equilibrium})$$

$$\mathbf{d)} \quad m = \int 4\pi r^2 \rho dr \quad (\text{conservation of mass})$$

Combining these equations, one obtains the TOV equation:

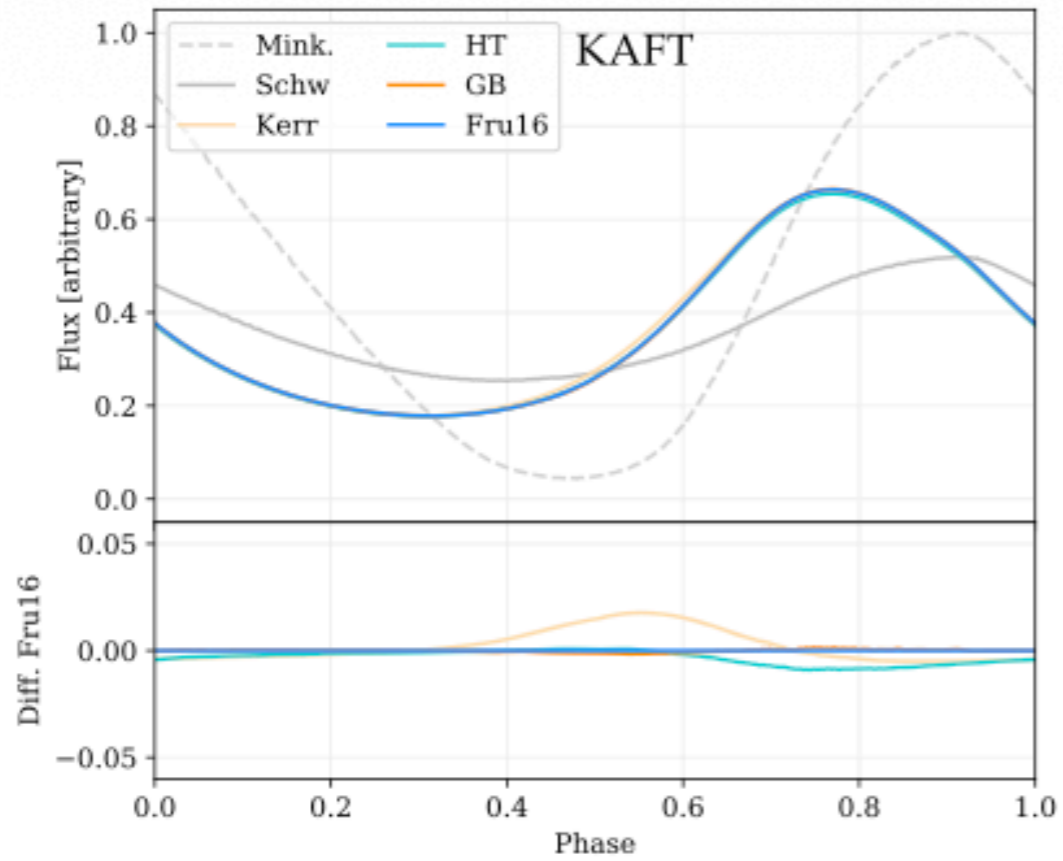
$$\frac{dp}{dr} = - (\rho + p) \frac{m + 4\pi r^3 p}{r(r - 2m)}.$$

Since m can be written in terms of ρ , we see that this equation has two unknown functions, $\rho(r)$ and $p(r)$. If we introduce an **equation of state**, $p(\rho)$, then we can reduce this equation to a differential equation with only one unknown function, that can be solved numerically. Once the pressure (or density) are solved, then we immediately get the remaining unknowns in the metric, $\Phi(r)$, $\Lambda(r)$. The TOV equation is the general relativistic equivalent to the first two stellar structure equations.



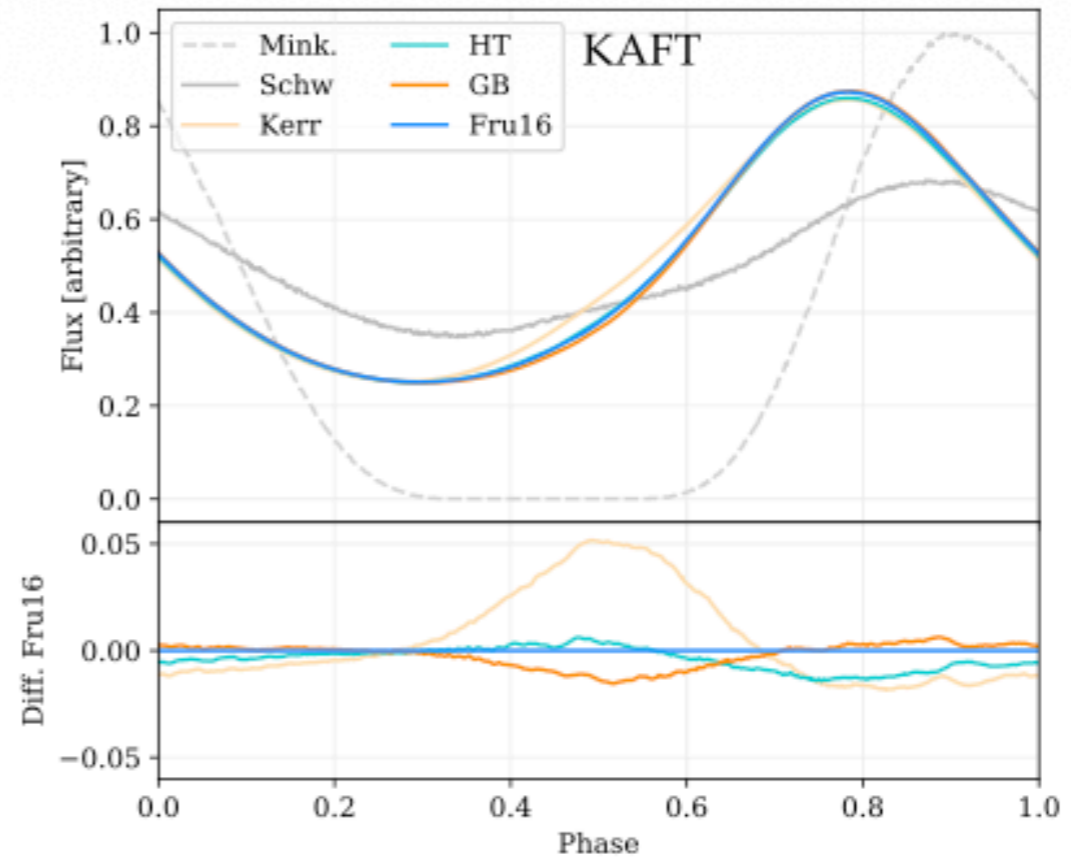
Equation of state and mass-radius curve

The left plot shows several calculations of the equation of state of a neutron star assuming different compositions. The right plot shows what the predicted masses and radii yield for different compositions. Except for the few gray curves, more massive a neutron star is, the smaller the radius.



(a)

View: $i = 30^\circ$; hot spot: circular $\Theta = 45^\circ$ $\zeta = 10^\circ$



(b)

*View: $i = 0^\circ$; hot spot: crescent
 $\Theta_c = 2.8394$ rad $\zeta_c = 0.46$ rad $\phi_{c0} = 0.0606$ rad
 $\Theta_s = 2.91$ rad $\zeta_s = 0.47$ rad $\phi_{s0} = 0$ rad*

Pulses of a pulsar

These plots show the pulses of a pulsar assuming different models for the spacetime (different line colors). The left and right panel show different shapes and locations of the hotspots in the neutron star.